MASTER OF COMPUTER APPLICATIONS

MCA - 103

MATHEMATICS - I

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CONTENTS

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Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No. :**	**01**
Lesson	**:**	**Algebra : Quadratic Equations & Determinants**		

1.0 OBJECTIVES :

In this lesson, you will be able to understand

* To solve quadratic equations.
* To solve equations reducible to quadratic.
* To solve simultanens linear & quadratic equations.
* Determinants, properties of determinants,
* Solution of linear equations using Cramer's rule.

1.1 INTRODUCTION :

A function 'f' defined by

$f(x) = a_0 + a_1x + a_2x^2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_nx^n$,

where $a_0, a_1, a_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots a_n \in R$

is called a polynomial in x with real coefficients, and if $a_n \neq 0$, then f(x) is said to be of degree n.

If $f(x) = a_0 + a_1x + a_2x^2 + a_nx^n = 0$ is a polynomial. of degree n (≥ 1),

then $f(x) = 0$

i.e., $a_0 + a_1x + a_2x^2 + a_nx^n = 0$

is called an **algebraic equation** of degree n.

A number α (real or complex) is called a root of the equation

$f(x) = a_0 + a_1x + a_2x^2 + a_nx^n = 0$

iff $f(\alpha) = 0$, i.e., iff $a_0 + a_1\alpha + a_2\alpha^2 + a_n\alpha^n = 0$

But here we deal with Quadratic Equations

Definition : An expression of the type $ax^2 + bx + c$,

$a, b, c \in R$, $a \neq 0$ is called a **quadratic polynomial** with real coefficients.

An **equation** of second degree $ax^2 + bx + c = 0$, where $a, b, c \in R$, $a \neq 0$ is called a **quadratic equation** with real coefficients.

eg., $5x^2 - 8x + 3 = 0$ and $3x^2 - 7 = 0$ are quadratic equations with real coefficients.

A number α (real or complex) is called root of the equation

$ax^2 + bx + c = 0$ iff $a\alpha^2 + b\alpha + c = 0$.

Thus, the roots of an equation are the numbers which when substituted in place of the **variable** in the given equation make its both sides equal. There are two types of quadratic equations viz.

(i) Pure Quadratic Equation : A quadratic equation is said to be pure quadratic if it contains no term of x (i.e., where Ist degree term is absent)

eg., $5x^2 - 7 = 0$, and

$7x^2 + 9 = 0$ are pure quadratic equations.

$\therefore$ $ax^2 + bx + c = 0$ is pure quadratic if b = 0.

(ii) Complete Quadratic Equation : A quadratic equation, $ax^2 + bx + c = 0$ is called a complete quadratic equation if a,b,c are not zero, e.g., $x^2-x+1=0$

Solution of Pure quadratic is of type

$ax^2 + c = 0, \ a \neq 0$

$ax^2 = -c$

$x^2 = -\frac{c}{a}$

$x = \pm\sqrt{-\frac{c}{a}}$

which are required solutions of the above pure quadratic equation.

1.2 METHOD OF SOLVING A QUADRATIC EQUATION :

1.2.1 Method of Factorisation

Here, we factorise the L.H.S. of equation and use the fact that if the product of two quantities is zero, then either of the two is zero.

Example 1. Solve the equation $4x^2 - 16x + 15 = 0$

Solution. The given equation is

$4x^2 - 16x + 15 = 0$

or $4x^2 - 10x - 6x + 15 = 0$

$\Rightarrow$ $(4x^2 - 10x) - (6x - 15) = 0$

$\Rightarrow$ $2x(2x - 5) - 3(2x - 5) = 0$

$\Rightarrow$ $(2x - 5)(2x - 3) = 0$

$\Rightarrow$ Either $2x - 5 = 0$ or $2x - 3 = 0$

$\therefore$ $x = \frac{5}{2}$ or $x = \frac{3}{2}$

$\therefore$ $x = 5/2, 3/2$ are the roots of given equation

1.2.2 Methods of Completing the Squares

To solve a quadratic equation, we transpose the constant term on the R.H.S. and divide both sides by the coefficient of x^2 to make it unity. Then, we make L.H.S. a perfect square by adding the square of half the coefficient of x to both sides. Finally, square roots of both sides will give the solution.

Example 2. Solve the equation $2x^2 - 7x + 6 = 0$

Solution : The given equation is

$2x^2 - 7x + 6 = 0$

or $2x^2 - 7x = -6$

or $\quad x^2 - \frac{7}{2}x + \frac{49}{16} = -3 + \frac{49}{16}$ $\quad$ $\left[\text{Adding } \left(\frac{7}{4}\right)^2 \text{ i.e. } \frac{49}{16}\right]$ on both sides

or $\quad \left(x - \frac{7}{4}\right)^2 = \frac{1}{16}$

$\Rightarrow \quad x - \frac{7}{4} = \pm \frac{1}{4}$

Either $\quad$ OR

$x - \frac{7}{4} = \frac{1}{4}$ $\quad$ | $\quad$ $x - \frac{7}{4} = -\frac{1}{4}$

$\Rightarrow \quad x = \frac{7}{4} + \frac{1}{4}$ $\quad$ | $\quad$ $x = \frac{7}{4} - \frac{1}{4}$

$\Rightarrow \quad x = 2$ $\quad$ | $\quad$ $\Rightarrow \quad x = \frac{3}{2}$

1.2.3. Solution with the help of Formula

To solve the quadratic equation

$ax^2 + bx + c = 0, \quad a \neq 0$

or $\quad ax^2 + bx = -c$

or $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}$

or $\quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

or $\quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

or $\quad \left(x + \frac{b}{2a}\right) = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$ $\quad$ or $\quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.3 EQUATIONS REDUCIBLE TO QUADRATIC : The following examples illustrate that the given equations can be reduced to quadratic by substitution :-

Example 3. Solve for x: $\frac{1}{p+q+x} = \frac{1}{p} + \frac{1}{q} + \frac{1}{x}$

Solution : The given equation can be written as

$$\frac{1}{p+q+x} - \frac{1}{x} = \frac{1}{p} + \frac{1}{q}$$

or $$\frac{-1}{x(p+q+x)} = \frac{1}{pq}$$

or $x(x+p+q) = -pq$

or $x^2 + x(p+q) + pq = 0$

which is a quadratic in x.

Comparing it with $ax^2 + bx + c = 0$, we have

$a = 1,\ b = p+q,\ c = pq$

Therefore, using formula we get

$$x = \frac{-(p+q) \pm \sqrt{(p+q)^2 - 4pq}}{2}$$

or $$x = \frac{-(p+q) \pm (p-q)}{2}$$

or $$x = -\frac{2q}{2} - \frac{2p}{2} = -q, -p$$

Hence roots are $x = -p, -q$.

Example. 4 Solve for x:

$2^{2x+3} - 57 = 65(2^x - 1)$

Solution : The given equation can be written as

$2^{2x}.2^3 - 57 = 65\,(2^x - 1)$ ———————— (1)

Put $2^x = y$, $\Rightarrow$ $2^{2x} = y^2$

Therefore, from (1), we get

or $8y^2 - 57 = 65\,(y - 1)$

or $8y^2 - 65y - 57 + 65 = 0$

or $8y^2 - 65y + 8 = 0$

or $8y^2 - 64y - y + 8 = 0$

or $8y\,(y - 8) - 1\,(y - 8) = 0$

$(y - 8)\ (8y - 1) = 0$

$\therefore$ Either $y - 8 = 0$ OR $8y - 1 = 0$

$y = 8$	or $y = \frac{1}{8}$
$2^x = 2^3$	i.e, $2^x = \left(\frac{1}{2}\right)^3 = 2^{-3}$
$\therefore\ x = 3$	$\therefore\ x = -3$

Example 5. Solve for x :

$2x^{1/3} + x^{2/3} - 8 = 0$ (1)

Solution : Put $x^{1/3} = y$ $\Rightarrow$ $x^{2/3} = y^2$ in (1), we get

$$y^2 + 2y - 8 = 0$$

or $y = \dfrac{-2 \pm \sqrt{4 + 32}}{2}$

or $y = \dfrac{-2 \pm 6}{2}$

or $y = 2, -4$

$\Rightarrow$ Either $y = 2$, or $y = -4$

i.e. $x^{1/3} = 2$

cubing both sides

$\therefore x = 8$

i.e. $x^{1/3} = -4$

cubing both sides,

$\therefore x = -64$

Example 4. Solve for x :

Solution : $10\sqrt{\dfrac{x}{x+3}} - \sqrt{\dfrac{x+3}{x}} = 3$

Putting $\sqrt{\dfrac{x}{x+3}} = y$ in the given equation, we get

or $10y - \dfrac{1}{y} = 3$

or $10y^2 - 1 = 3y$ or $10y^2 - 3y - 1 = 0$

using formula we get

$$y = \frac{3 \pm \sqrt{9+40}}{20}$$

$$= \frac{3 \pm 7}{20} = \frac{1}{2}, -\frac{1}{5}$$

$\Rightarrow$ Either $y = \dfrac{1}{2}$

i.e., $\sqrt{\dfrac{x}{x+3}} = \dfrac{1}{2}$

or $\dfrac{x}{x+3} = \dfrac{1}{4}$

or $4x = x + 3$

or $3x = 3$

or $x = 1$

or $y = -\dfrac{1}{5}$

i.e., $\sqrt{\dfrac{x}{x+3}} = -\dfrac{1}{5}$

or $\dfrac{x}{x+3} = \dfrac{1}{25}$

$25x = x + 3$

$24x = 3$

$x = \dfrac{1}{8}$

Rejected, as $\sqrt{\dfrac{x}{x+3}}$ should be

+ve real no.

If one or both of them are negative, we would have rejected the negative value (s)

Example. 7 Solve for x :

$$\sqrt{x + 5} + \sqrt{x + 12} = \sqrt{2x + 41}$$

Solution Squaring both sides of the given equation, we get

$$x + 5 + x + 12 + 2\sqrt{x + 5}\sqrt{x + 12} = 2x + 41$$

$$2\sqrt{x + 5}\sqrt{x + 12} = 41 - 17 = 24$$

or $\sqrt{x + 5}\ \sqrt{x + 12} = 12$

Again squaring both sides,

$(x + 5)(x + 12) = 144$

or $x^2 + 17x + 60 - 144 = 0$

or $x^2 + 17x - 84 = 0$

or $(x + 21)(x - 4) = 0$

$\therefore$ $x = -21, 4$

But $x = -21$ does not satisfy the given equation. Hence, root of the given equation is x=4

Example. 8 Solve :

$$\left(x + \frac{1}{x}\right)^2 - 2\left(x - \frac{1}{x} + 4\right) - 11 = 0$$

Solution : The given equation can be written as

$$\left(x + \frac{1}{x}\right)^2 - 2\left(x - \frac{1}{x}\right) - 8 - 11 = 0$$

or $x^2 + \frac{1}{x^2} + 2 - 2\left(x - \frac{1}{x}\right) - 19 = 0$

or $\left(x^2 + \frac{1}{x^2}\right) - 2\left(x - \frac{1}{x}\right) - 17 = 0$

or $\left[\left(x - \frac{1}{x}\right)^2 + 2\right] - 2\left(x - \frac{1}{x}\right) - 17 = 0$

Put $x - \frac{1}{x} = y$, we have

$(y^2 + 2) - 2y - 17 = 0$

or $y^2 - 2y - 15 = 0$

$y^2 - 5y + 3y - 15 = 0$ or $(y - 5)(y + 3) = 0$

$\Rightarrow$ Either $y - 5 = 0$

or $y = 5$

i.e., $x - \frac{1}{x} = 5$

or $x^2 - 1 = 5x$

or $x^2 - 5x - 1 = 0$

or $x = \frac{5 \pm \sqrt{25 + 4}}{2}$

$x = \frac{5 \pm \sqrt{29}}{2}$

OR $y + 3 = 0$

or $y = -3$

i.e., $x - \frac{1}{x} = -3$

or $x^2 - 1 = -3x$

$x^2 + 3x - 1 = 0$

or $x = \frac{-3 \pm \sqrt{9 + 4}}{2}$

$x = \frac{-3 \pm \sqrt{13}}{2}$

Example. 9 Solve :

$3x^2 - 7 + 3\sqrt{3x^2 - 16x + 21} = 16x$

Solution : The given equation can be written as

$3x^2 - 16x - 7 + 3\sqrt{3x^2 - 16x + 21} = 0$ ———— (1)

Put $3x^2 - 16x + 21 = y$

(1)$\Rightarrow$ $y - 28 + 3\sqrt{y} = 0$

or $y - 28 = -3\sqrt{y}$

Squaring both sides,

$y^2 - 56y + 784 = 9y$

or $y^2 - 65y + 784 = 0$

or $y^2 - 49y - 16y + 784 = 0$

$(y - 49)(y - 16) = 0$

$\Rightarrow$ Either $y = 49$ or $y = 16$

$y = 49$

i.e., $3x^2 - 16x + 21 = 49$

or $3x^2 - 16x - 28 = 0$

or $x = \dfrac{16 \pm \sqrt{256 + 336}}{6}$

$x = \dfrac{16 \pm \sqrt{592}}{6}$

OR

$y = 16$

i.e., $3x^2 - 16x + 21 = 16$

or $3x^2 - 16x + 5 = 0$

or $(3x - 1)(x - 5) = 0$

$x = \dfrac{1}{3}$, or $x = 5$.

Hence, roots of given equation are $\dfrac{1}{3}$, 5, $\dfrac{16 \pm \sqrt{592}}{6}$

Example 10. Find the value of

$$2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots\dots\infty}}$$

Solution :

Let $x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \dots\dots\infty}}}$

Clearly we see that the expression below the first horizontal line is also equal to x, because like the original expression it also begins with 2 and goes on similarly to infinity.

Therefore, $x = 2 + \frac{1}{x}$

or $x^2 = 2x + 1$

or $x^2 - 2x - 1 = 0 \quad \Rightarrow \quad x = \frac{2 \pm \sqrt{4 + 4}}{2}$

or $x = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$

But the given expression is essentially +ve. Hence, rejecting the negative value, we get

$x = 1 + \sqrt{2}$

Example. 11 Find the value of $\sqrt{\mathbf{20 + \sqrt{20 + \sqrt{20 + ... \infty}}}}$

Solution:Let $x = \sqrt{20 + \sqrt{20 + \sqrt{20 + ... \infty}}}$

Then $x = \sqrt{20 + x}$

Squaring both sides, we get

$x^2 = 20 + x$

or $x^2 - x - 20 = 0$

or $(x - 5)(x + 4) = 0$

or $x = 5, \; x = -4$

Rejecting negative value, we have $x = 5$

1.4 SIMULTANEOUS LINEAR EQUATIONS :

When an equation contains more than one variable, it is called a simultaneous equation.

e.g., $ax + by + c = 0, (a \neq 0, b \neq 0)$

where a,b,c are constants is called a linear equation in two variables x and y. To find the solution of simultaneons equations, we will be given as many number of equations as the number of variables we consider.

Example. 12 Solve for x and y :

$$5x + 2y = 12 \quad \text{———} (1)$$

$$2x^2 + 3xy + y^2 = 15 \quad \text{———} (2)$$

Solution : From linear equation (1), 5x+2y = 12,

we have

$$x = \frac{12 - 2y}{5}$$

substitute this value in given quadratic equation (2), we get

$$2\left(\frac{12 - 2y}{5}\right)^2 + 3y\left(\frac{12 - 2y}{5}\right) + y^2 = 15$$

simplifying, we obtain $y^2 + 28y - 29 = 0$

Solving, y = 1 & y = -29

Substituting these values in linear equation we get

when y = 1, x = 2	when y = -29 then x = 14

Example. 13 Solve for x and y : x + y = 17, xy = 60

Solution : we know that $(x - y)^2 = (x + y)^2 - 4xy$

$$= 289 - 240 = 49$$

$\Rightarrow x - y = \pm 7$

Thus, we get the equations

x + y = 17 x - y = 7	and x + y = 17 x - y = -7

Solving these,we get

x = 12, y = 5 and x = 5, y = 12

Example. 14 Solve the equations:

$x + y = 5, \quad x^2 + y^2 = 13$

Solution : We have $2xy = (x + y)^2 - (x^2 + y^2)$

or $2xy = 25 - 13 = 12$

or $xy = 6$

Now solve: $x + y = 5$, $xy = 6$

Using $(x - y)^2 = (x + y)^2 - 4xy$

$= 25 - 24 = 1$

$x - y = \pm 1$

Therefore, $x + y = 5$,

$x - y = 1$

Solving these, we get

$x = 3$

& $y = 2$

and $x + y = 5$

$x - y = -1$

Solving these, we get

$x = 2$ & $y = 3$

Example. 15 Solve the equations:

$4x^2 - y^2 = 96, \quad 2x - y = 8$

Solution : Dividing the first equation by the corresponding sides of second equation,

$$\frac{4x^2 - y^2}{2x - y} = \frac{96}{8}$$ we get

$2x + y = 12$

Now solve $2x + y = 12$ & $2x - y = 8$

we get

$x = 5, \quad y = 2$

1.5 DETERMINANTS :

Let us eliminate x and y from the equations

$$a_1x + b_1y = 0 \quad \text{and} \quad a_2x + b_2y = 0$$

The eliminant is $\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$ $\qquad$ (each = $-\dfrac{y}{x}$)

or $\qquad a_1b_2 - a_2b_1 = 0$

which is conveniently written in the form $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

The expression $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called a **determinant of the second order.** The numbers a_1, b_1, a_2, b_2 are called the **elements** of the determinant & $a_1b_2 - a_2b_1$ is called the **expansion** or the **value** of the determinant.

A determinant of the second order has 2^2 elements arranged in the form of a square in 2 horizontal lines (called **rows**) and 2 vertical lines (called **columns**) and is bounded by two vertical lines.

Let us eliminate x, y and z from the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z = 0 \\ a_2x + b_2y + c_2z = 0 \\ a_3x + b_3y + c_3z = 0 \end{aligned}\right\} \quad \text{........(1)}$$

Solving last two equations of the above equation (1) we get

$$\frac{x}{b_2c_3 - b_3c_2} = \frac{y}{c_2a_3 - c_3a_2} = \frac{z}{a_2b_3 - a_3b_2}$$

Substituting these proportional values of x, y and z in first equation of (1), we get, $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$

which is conveniently written in the form $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = 0$

i.e., $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$..................(2)

The expression $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is called a **determinant of third order**

the **expansion** or the **value** of the determinant is given on R.H.S. of (2)

We note that a determinant of the third order has 3^2 elements arranged in the form of a square along 3 horizontal lines (called **rows**) and 3 vertical lines (called **columns**) and is bounded by two vertical lines.

Similarly, A determinant of the *n*th order is denoted by

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13}.....a_{1n} \\ a_{21} & a_{22} & a_{23}.....a_{2n} \\ .... \\ .... \\ a_{n1} & a_{n2} & a_{n3}....a_{nn} \end{vmatrix}$$

which is a block of n^2 elements arranged in the form of a square along 'n' horizontal lines (called **rows**) and 'n' vertical lines (called **columns**) and is bounded by two vertical lines.

In a_{ij}, the first suffix i indicates the row and the second suffix j indicates the column in which the element lies. i.e., a_{ij} is the element in the ith row and jth column.

The elements $a_{11}, a_{22}, a_{33}, ..., a_{nn}$ (i.e. the elements a_{ij} with i = j) are called the **diagonal elements.** The diagonal through the left hand top corner along which the diagonal elements lie is called the **leading or principal diagonal.** Also $a_{11}\ a_{22}\ a_{33}$ .... a_{nn} is called the **leading term.**

1.5.1 MINORS AND CO-FACTORS

The **minor** of an element in a determinant is the determinant obtained by omitting the row and the column in which that element lies.

For example, consider the determinant, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

To find the minor of c_2 which occur in the second row and third column, we omit the second row and the third column.

Thus, minor of $c_2 = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

Similarly, minor of $a_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

The **co-factor** of an element in a determinant is its minor with proper sign and is usually denoted by the corresponding capital letter. The co-factor of the element which lies in the ith row and jth column is $(-1)^{i+j}$ times the minor of the element.

Thus, the co-factor of $c_2 = (-1)^{2+3} \times$ minor of $c_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = C_2$

Similarly, A_3 = co-factor of $a_3 = (-1)^{3+1} \times$ minor of $a_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

1.5.2 EXPANSION OF A DETERMINANT

(i) For a determinant of a second order, we have $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$

sign of a product remains unchanged with downward arrow while it changes with upward arrow.

$\Rightarrow \begin{vmatrix} -2 & 5 \\ -3 & 4 \end{vmatrix} = (-2)(4) - (-3)(5) = 7$

(ii) For a determinant of third order, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$

$$= a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1A_1 + b_1B_1 + c_1C_1$$

= the sum of the products of elements in the first row with their respective co-factors

Example Find the value of the determinant $D = \begin{vmatrix} 2 & -4 & 3 \\ 3 & 1 & 2 \\ 7 & 6 & 1 \end{vmatrix}$ by expanding from second row

Solution Expanding from second row,

$$D = 3\left(-\begin{vmatrix} -4 & 3 \\ 6 & 1 \end{vmatrix}\right) + 1\begin{vmatrix} 2 & 3 \\ 7 & 1 \end{vmatrix} + 2\left(-\begin{vmatrix} 2 & -4 \\ 7 & 6 \end{vmatrix}\right)$$

$$= -3\ (-4-18) + (2-21) -2\ (12-28) = 66 - 19 - 80 = -33$$

1.5.3 PROPERTIES OF DETERMINANTS

The following properties of determinants simplify their evaluation. We shall verify these properties for determinants of third order only, however, they hold good for determinants of any order.

Property I *The value of a determinant remains unaltered if rows & columns are interchanged. This is always denoted by ' & is also called* ***transpose****.*

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let D' be the determinant obtained from D by changing its rows into columns and columns into rows, then

$$D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Expanding D by first row and D' by first column, each is equal to

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$

Property II *If Any two rows (or columns) of a determinant be interchanged, the determinant is unaltered in numerical value, but changed in sign only.*

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let D' be the determinant obtained from D by interchanging its second and third rows, then

$$D' = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

Then $D' = -D$

Property III *If a determinant has two rows (or columns) identical, then its value is zero.*

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Then $D = 0$

Property IV *If all the elements of any row (or culumn) of a determinant are multiplied by a common number, then determinant is multiplied by that number.*

Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Let D' be the determinant obtained from D by multiplying each element of first row (say) by k, then

$D' = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then, $D' = kD$

Property V *If each element of any row (or column) can be expressed as the sum of two terms, then the determinant can be expressed as the sum of two determinants.*

Consider $$D = \begin{vmatrix} a_1+a_1' & b_1+b_1' & c_1+c_1' \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

where each element of first row is the sum of two terms. Then

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1' & c_1' \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

thus, D has been expressed as the sum of two determinants of the third order.

Property VI *The value of a determinant is not altered by adding to the elements of any row (or column) the same multiples of the corresponding elements of any other row (or column).*

Let $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Let D' be the determinant obtained from D by adding to each element of first column p times the corresponding elements of second column and q times the corresponding elements of third column ; then

$$D' = \begin{vmatrix} a_1+pb_1+qc_1 & b_1 & c_1 \\ a_2+pb_2+qc_2 & b_2 & c_2 \\ a_3+pb_3+qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qc_1 & b_1 & c_1 \\ qc_2 & b_2 & c_2 \\ qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= D + p\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} + q\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}$$

$$D' = D + p(0) + q(0) = D$$

since two columns of second as well as third determinants are identical.

Note : It should be noted that while applying Prop.VI atleast ONE row (or column) must remain unchanged.

Property VII *If a determinant D vanishes for x = a, then (x-a) is a factor of D. In other words, if two rows (or two columns) become identical for x = a, then (x-a) is a factor of D.*

In general, if 'r' rows (or 'r' columns) become identical when 'a' is substituted for x, then $(x-a)^{r-1}$ is a factor of D.

Example. 16 Show that $\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$

Solution. Interchanging rows & columns,

$$\text{L.H.S.} = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} a & x & p \\ b & y & q \\ c & z & r \end{vmatrix} = (-1)\begin{vmatrix} x & a & p \\ y & b & q \\ z & c & r \end{vmatrix} \quad \text{Interchanging } C_1, C_2$$

$$= (-1)^2 \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} \qquad \text{Interchanging } R_1 \ \& \ R_2$$

$$= \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix}$$

Example. 17 If $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0$ in which a, b, c are different, show that abc = 1.

Solution. In the given determinant on L.H.S., each element of C_3 is the sum of two terms, therefore, we can express it as the sum of two determinants.

therefore, $D = \begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix}$

Taking out a, b, c common from R_1, R_2, R_3 respectively in the first determinant and -1 from C_3 in the second determinant

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

Changing C_3 over C_2 and C_1 in the second determinant

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - (-1)^2 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (abc - 1) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Operating $R_2 - R_1$ and $R_3 - R_1$

$$= (abc - 1) \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \qquad \text{(Expanding by } C_1\text{)}$$

$$= (abc - 1) \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

Taking out b - a, c - a common from R_1 and R_2 respectively

$$= (abc - 1)(b - a)(c - a) \begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$$= (abc - 1)(b - a)(c - a)[(c + a) - (b + a)] = (abc - 1)(b - a)(c - a)(c - b)$$

Now $D = 0$

$\Rightarrow$ $(abc - 1)(b - a)(c - a)(c - b) = 0$

$\Rightarrow$ $abc - 1 = 0$ since $a \neq b \neq c$

$\Rightarrow$ $abc = 1$

Example. 18 Without expanding the determinant, prove that

$$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$

Solution. Let $D = \begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix}$ take -1 common from C_1, C_2, C_3

therefore $D = (-1)(-1)(-1)\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix}$

$= -D$ therefore $D + D = 0 \quad \Rightarrow D = 0$

Example. 19 Prove that $\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$

Solution. Denoting the given determinant by D and operating C_1-C_3, C_2-C_3, we have

$$D = \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c+a)(b+c-a) & 0 & a^2 \\ 0 & (c+a+b)(c+a-b) & b^2 \\ (c+a+b)(c-a-b) & (c+a+b)(c-a-b) & (a+b)^2 \end{vmatrix}$$

Taking out (a+b+c) from C_1 and C_2

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix} \quad \text{Operaring } R_3\text{-}R_2\text{-}R_1$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix} \quad \text{Operaring } C_1 + \frac{1}{a}C_3,\ C_2 + \frac{1}{b}C_3$$

$$= (a+b+c)^2 \begin{vmatrix} b+c & \frac{a^2}{b} & a^2 \\ \frac{b^2}{a} & c+a & b^2 \\ 0 & 0 & 2ab \end{vmatrix} \quad \text{Expanding along } R_3$$

$$= 2ab\,(a+b+c)^2 \begin{vmatrix} b+c & \dfrac{a^2}{b} \\ \dfrac{b^2}{a} & c+a \end{vmatrix}$$

$$= 2ab(a+b+c)^2\,(bc+ab+c^2+ca-ab)$$

$$= 2ab(a+b+c)^2\,(bc+c^2+ca) \;=\; 2abc(a+b+c)^3$$

Example 20 Prove that $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}\right)$

Solution. Denoting the given determinant by D, we get

$$D = \begin{vmatrix} a\left(\dfrac{1}{a}+1\right) & a\,.\dfrac{1}{a} & a\,.\dfrac{1}{a} & a\,.\dfrac{1}{a} \\ b\,.\dfrac{1}{b} & b\left(\dfrac{1}{b}+1\right) & b\,.\dfrac{1}{b} & b\,.\dfrac{1}{b} \\ c\,.\dfrac{1}{c} & c\,.\dfrac{1}{c} & c\left(\dfrac{1}{c}+1\right) & c\,.\dfrac{1}{c} \\ d\,.\dfrac{1}{d} & d\,.\dfrac{1}{d} & d\,.\dfrac{1}{d} & d\left(\dfrac{1}{d}+1\right) \end{vmatrix}$$

Taking out a, b, c, d common from R_1, R_2, R_3, R_4, respectively

$$= abcd \begin{vmatrix} 1+\dfrac{1}{a} & \dfrac{1}{a} & \dfrac{1}{a} & \dfrac{1}{a} \\ \dfrac{1}{b} & 1+\dfrac{1}{b} & \dfrac{1}{b} & \dfrac{1}{b} \\ \dfrac{1}{c} & \dfrac{1}{c} & 1+\dfrac{1}{c} & \dfrac{1}{c} \\ \dfrac{1}{d} & \dfrac{1}{d} & \dfrac{1}{d} & 1+\dfrac{1}{d} \end{vmatrix}$$

Operating $R_1+R_2+R_3+R_4$ and taking out $\left(1+\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}\right)$ from R_1

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} & \frac{1}{c} \\ \frac{1}{d} & \frac{1}{d} & \frac{1}{d} & 1+\frac{1}{d} \end{vmatrix}$$

Operating $C_2 - C_1$, $C_3 - C_1$, $C_4 - C_1$

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{b} & 1 & 0 & 0 \\ \frac{1}{c} & 0 & 1 & 0 \\ \frac{1}{d} & 0 & 0 & 1 \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

since the value of the triangular determinant is 1, the product of diagonal elements.

1.5.4 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS (CRAMER'S RULE)

Let the system of linear equations in three unknowns x, y, z be

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \qquad \text{.................(1)}$$

Also, let the determinant of co-efficients be

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$$

then $$x.D = x\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1x & b_1 & c_1 \\ a_2x & b_2 & c_2 \\ a_3x & b_3 & c_3 \end{vmatrix}$$

Operating $C_1 + yC_2 + zC_3$, we have

$$xD = \begin{vmatrix} a_1x + b_1y + c_1z & b_1 & c_1 \\ a_2x + b_2y + c_2z & b_2 & c_2 \\ a_3x + b_3y + c_3z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} = D_1 \text{ (say)}$$

Similarly, $yD = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} = D_2$ (say) and $zD = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = D_3$ (say)

Thus, we have $xD = D_1, yD = D_2, zD = D_3$

Since $D \neq 0$, the system has unique solution, given by $x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$

Note 1 D_1, D_2, D_3 are the determinants obtained from D by replacing the elements of columns C_1, C_2, C_3 respectively by d_1, d_2, d_3.

Note 2 The system of equations has a unique solution if $D \neq 0$

Note 3 If $D = 0$ and at least one of the three determinants D_1, D_2, D_3 is non-zero , then the system of equations has no solution

Note 4. If $D = 0$ and also $D_1 = D_2 = D_3 = 0$, then the system of equations has an infinite number of solutions

Example. 21 Solve by Cramer's Rule : $2x + 3y - 7 = 0$
$x + 2y - 4 = 0$

Solution : We have

$$D = \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 3 = 1$$

$$D_1 = \begin{vmatrix} 7 & 3 \\ 4 & 2 \end{vmatrix} = 14 - 12 = 2$$

$$D_2 = \begin{vmatrix} 2 & 7 \\ 1 & 4 \end{vmatrix} = 8 - 7 = 1$$

$$x = \frac{D_1}{D} = \frac{2}{1} = 2,$$

$$y = \frac{D_1}{D} = \frac{1}{1} = 1,$$

Example. 22 Solve by Cramer's Rule : $x - 3y + z = 2$

$3x + y + z = 6$

$5x + y + 3z = 3$

Solution. Here $D = \begin{vmatrix} 1 & -3 & 1 \\ 3 & 1 & 1 \\ 5 & 1 & 3 \end{vmatrix}$ (Expanding along R_1)

$= 1(3-1) + 3(9-5) + 1(3-5) = 12 \neq 0$

therefore, The system has a unique solution.

$D_1 = \begin{vmatrix} 2 & -3 & 1 \\ 6 & 1 & 1 \\ 3 & 1 & 3 \end{vmatrix}$ (Expanding along R_1)

$= 2(3-1) + 3(18-3) + 1(6-3) = 52$

Similarly, $D_2 = \begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 5 & 3 & 3 \end{vmatrix} = -14$

$D_3 = \begin{vmatrix} 1 & -3 & 2 \\ 3 & 1 & 6 \\ 5 & 1 & 3 \end{vmatrix} = -70$

therefore, $x = \frac{D_1}{D} = \frac{52}{12} = \frac{13}{3}$, $y = \frac{D_2}{D} = \frac{-14}{12} = \frac{-7}{6}$, $z = \frac{D_3}{D} = \frac{-70}{12} = \frac{-35}{6}$

Hence the solution of the given system is $x = \frac{13}{3}$, $y = \frac{-7}{6}$, $z = \frac{-35}{6}$

Example 23. Solve by Crames's rule :
$6x+y-3z=5, x+3y-2z=5, 2x+y+4z = 8$

Solution : Here

$D = \begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix} = 91 \neq 0$

Thus the given system has a unique solution

$$D_1 = \begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix} = 91, \quad D_2 = \begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix} = 182, \quad D_3 = \begin{vmatrix} 6 & 1 & 5 \\ 1 & 3 & 5 \\ 2 & 1 & 8 \end{vmatrix} = 91,$$

Therefore, $x = \frac{D_1}{D} = \frac{91}{91} = 1;\ y = \frac{D_2}{D} = \frac{182}{91} = 2;\ z = \frac{D_3}{D} = \frac{91}{91} = 1;$

Hence $x=1, y=2, z=1,$

1.6 SELF ASSESSMENT QUESTIONS :

Solve the following equations :

(1) $8x^2-2x-3=0$

(2) $4^x-5.2^x+4=0$

(3) $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$

(4) $2\left(x^2+\frac{1}{x^2}\right) - \left(x+\frac{1}{x}\right) - 11 = 0$

(5) $3x^2 - 2x - \sqrt{3x^2-2x+4} = 16$

(6) $2x+3y = 10,\ 3x+2y = 10$

(7) $x+y=5,\ x^2+2y^2=17$

(8) Without expanding, prove that the following determinate vanish.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(9) Prove without expanding that

$$\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$$

(10) Prove that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2\,(a+b+c)^3$$

(11) Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

(12) Solve the following system of equations, using determinants

$$5x-10y = 4$$
$$x - 2y = 8$$

(13) Solve using Cramer's rule

$$x+y+z = -1,\ x+2y+3z = -4,\ x+3y+4z = -6$$

(14) Solve for x and y : $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 5/2,\ x+y = 10$

1.7 KEY WORDS :

Quadratic equation, Simultaneous equations, Determinants, Cramer's rule.

1.8 SUGGESTED READINGS :

1. Grewal, B.S. - Engineering Mathematics
2. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.
3. Ramana, B.V. - Higher Engineering Mathematics.
4. Aggarwal, R.S. - Modern Approach of Mathematics.

Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No. :**	**02**
Lesson	**:**	**Algebra - Matrices**		

2.0 OBJECTIVES :

In this lesson, you will be able to understand

* Types of matrices, scalar multiplication of a matrix, equality of matrices.
* Basic laws of matrix algebra, namely addition, subtraction, multiplication.
* Transpose of a matrix, adjoint of a matrix, inverse of a matrix.
* Solution of equations by matrix method.

2.1 INTRODUCTION :

A set of "mn" numbers (real or comple) arranged in a rectangular array having 'm' rows (horizontal lines) and 'n' columns (vertical lines), the numbers being enclosed by brackets [ ] or ( ), is called an "m × n" **matrix** (read as "m by n matrix").

An "m × n" matrix is also called a matrix of order m × n. Each of mn numbers is called an **element** of the matrix.

For example, $\begin{bmatrix} 5 & -1 & 2 \\ 6 & 0 & 4 \end{bmatrix}$ is a 2 × 3 matrix or matrix of order 2 × 3.

It has two rows and three columns. The numbers 5, -1, 2, 6, 0, 4 are its elements.

2.2 DEFINITIONS :

2.2.1 Matrix :

An "m × n" matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots & a_{2n} \\ a_{31} & a_{32} & a_{33} \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} \ldots & a_{mn} \end{bmatrix}$$

Here each element has two suffixes. The first suffix indicates the row and the second suffix indicates the column in which the element lies. Thus, a_{12} is the element lying in the First row and Second column, a_{ij} is the element lying in the ith row and jth column.

A matrix is usually denoted by a single capital letter A or B or C etc.

Thus, an m × n matrix A may be written as

$$A = [a_{ij}]_{m\times n} \text{ or } A = [a_{ij}], \text{ where } i = 1,2,3,\ldots\ldots,m, j = 1,2,3,\ldots\ldots,n$$

2.2.2 Square Matrix : A matrix in which the number of rows is equal to the number of columns is called a **square matrix,** otherwise, it is said to be a **rectangular matrix.**

Thus, a matrix $A = [a_{ij}]_{m\times n}$ is a square matrix if m = n and a rectangular matrix if $m \neq n$.

A square matrix having n rows and n columns is called "a square matrix of order n" or "an n-rowed square matrix",

e.g. $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & b_{32} & c_{33} \end{bmatrix}$ is a square matrix of order three

The elements a_{11}, a_{22}, a_{33} of a square matrix are called its **diagonal elements** and the diagonal along which these elements lie is called a **principal diagonal.**

In a square matrix $A = [a_{ij}]$,

(i) for elements along the principal diagonal, $i = j$

(ii) for elements above the principal diagonal, $i < j$

(iii) for elements below the principal diagonal, $i > j$

(iv) for non-diagonal elements, $i \neq j$

The sum of the diagonal elements of a square matrix is called its **trace.**

Trace of a square matrix $A = [a_{ij}]$ of order n is, $a_{11}+a_{22}+a_{33}+\dots\dots\dots\dots+a_{nn} = \sum_{i=1}^{n} a_{ii}$

2.2.3 Row Matrix : A matrix having only one row and any number of columns i.e. a matrix of order $1 \times n$ is called a **row matrix**. e.g. $[3 \quad 5 \quad -7 \quad 2]$ is a row matrix.

2.2.4 Column Matrix : A matrix having only one column and any number of rows i.e. a matrix of order $m \times 1$ is called a **column matrix.** e.g. $\begin{bmatrix} 5 \\ 0 \\ -2 \end{bmatrix}$ is a column matrix.

2.2.5 Null Matrix: A matrix in which each element is zero is called a **null matrix** or a **zero matrix**. A null matrix of order $m \times n$ is denoted by $O_{m\times n}$.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{3\times 2} \ , \ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = O_{2\times 4}$$

2.2.6 Sub-Matrix: A matrix obtained from a given matrix A by deleting some of its rows or column or both is called a **sub-matrix of A.**

Thus, $B = \begin{bmatrix} 7 & 0 \\ 2 & 1 \end{bmatrix}$ is a sub-matrix of $A = \begin{bmatrix} 0 & -4 & 2 & 1 \\ 7 & 5 & 0 & 9 \\ 2 & 6 & 1 & -2 \end{bmatrix}$

obtained by deleting the first row, second and fourth columns of A.

2.2.7 Diagonal Matrix. A square matrix in which all non-diagonal elements are zero is called a **diagonal matrix.**

Thus, $A = [a_{ij}]_{n\times n}$ is a diagonal matrix if $a_{ij} = 0$ for $i \neq j$

For example, $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a diagonal matrix.

An n-rowed diagonal matrix is breifly written as diag. $[d_1, d_2, \ldots\ldots\ldots, d_n]$, where $d_1, d_2, \ldots\ldots\ldots, d_n$ are the diagonal elements. Thus, the above diagonal matrix A can be written as diag. [5, 0, 2].

2.2.8 Scalar Matrix. A diagonal matrix in which all the diagonal elements are equal to a scalar, say k, is called a scalar matrix, ie., A square matrix in which all the non-diagonal elements are zero and all diagonal elements are equal is called a scalar matrix.

i.e., $A = [a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = 0$ when $i \neq j$ & $a_{ij} = k$ when $i = j$

For example, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$ are scalar matrices.

2.2.9 Unit Matrix or Identity Matrix. A scalar matrix in which each diagonal element is unity (i.e. 1) is called a **unit matrix or identity matrix.**

Thus, a unit matrix is a square matrix in which non-diagonal elements are zero and all diagonal elements are equal to 1.

i.e., $A = [a_{ij}]_{n\times n}$ is a unit matrix if $a_{ij} = 0$ when $i \neq j$ & $a_{ij} = 1$ when $i = j$

A unit matrix of order n is denoted by I_n. or it may be simply denoted by I.

Thus, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

2.2.10 Triangular Matrix. A square matrix in which all the elements either below or above the principal diagonal are zero is called a triangular matrix.

2.2.11 Upper Triangular Matrix. A square matrix in which all the elements **below** the principal diagonal are zero is called an **upper triangular matrix.**

Thus, $A = [a_{ij}]_{n \times n}$ is a upper triangular matrix if $a_{ij} = 0$ for $i > j$.

For example, $\begin{bmatrix} 3 & 2 & -4 \\ 0 & 1 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ is an upper triangular matrix.

2.2.12 Lower Triangular Matrix. A square matrix in which all the elements **above** the principal diagonal are zero is called a **lower triangular matrix.**

Thus, $A = [a_{ij}]_{n \times n}$ is a lower triangular matrix if $a_{ij} = 0$ for $i < j$.

For example, $\begin{bmatrix} -4 & 0 & 0 \\ 3 & 2 & 0 \\ 5 & 7 & 0 \end{bmatrix}$ is a lower triangular matrix.

2.2.13 Equal Matrices. Two matrices A and B are said to be equal (written as A=B) if only if they have the same order and their corresponding elements are equal.

Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ then A = B if and only if

(i) m=p and n=q (ii) $a_{ij}=b_{ij}$ for all i and j.

2.3 BASIC OPERATIONS ON MATRICES :

2.3.1 ADDITION OF MATRICES

Two matrices are said to be conformable for addition if they have the same order.

If A and B are two matrices of the same order, then their sum A+B is a matrix each element of which is obtained by adding the corresponding elements of A and B.

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then $A+B=C=[c_{ij}]_{m\times n}$, where $c_{ij}=a_{ij}+b_{ij}$

Similarly, if A and B are two matrices of the same order, then their difference A-B is a matrix whose elements are obtained by substracting the elements of B from the corresponding elements of A.

In general, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then $A-B=C=[c_{ij}]_{m\times n}$, where $c_{ij}=a_{ij}-b_{ij}$

For example, if $A = \begin{bmatrix} 2 & 5 & -1 \\ 3 & 0 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -6 & 2 \\ -2 & 5 & 7 \end{bmatrix}$

then $$A+B = \begin{bmatrix} 2+1 & 5-6 & -1+2 \\ 3-2 & 0+5 & 4+7 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 5 & 11 \end{bmatrix}$$

and $$A-B = \begin{bmatrix} 2-1 & 5-(-6) & -1-2 \\ 3-(-2) & 0-5 & 4-7 \end{bmatrix} = \begin{bmatrix} 1 & 11 & -3 \\ 5 & -5 & -3 \end{bmatrix}$$

2.3.2 MULTIPLICATION OF A MATRIX BY A SCALAR

The product of matrix $A = [a_{ij}]$ by a scalar k is denoted by kA and is obtained by multiplying every element of A by k.

Thus, $kA = [ka_{ij}]$

If $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$, then $kA = \begin{bmatrix} ka_1 & ka_2 & ka_3 \\ kb_1 & kb_2 & kb_3 \end{bmatrix}$

In particular, $-A = (-1)\ A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ -b_1 & -b_2 & -b_3 \end{bmatrix}$

2.3.3 PROPERTIES OF MATRIX ADDITION

(i) **Matrix addition is commutative,** i.e., A+B = B+A

(ii) **Matrix addition is associative,** i.e., (A+B) + C = A + (B+C)

(iii) For any matrix A, there exists a null matrix O of the same order as A such that A+O = O+A = A.

(iv) For any matrix A, there exists a matrix -A of the same order as A such that

$$A + (-A) = (-A) + A = 0$$

Example 1. Find x, y, z and w if $3\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$

Sol. The give equation is $\begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$

Equating the corresponding elements on the two sides

$3x = x+4$, $3y = 6+x+y$, $3z = -1+z+w$, $3w = 2w + 3$

$\Rightarrow$ $2x = 4$, $2y = 6 + x$, $2z = -1 + w$, $w = 3$

$\Rightarrow$ $x = 2$, $y = 4$, $z = 1$, $w = 3$.

2.3.4 MATRIX MULTIPLICATION

Two matrices A and B are said to be conformable for the product AB **(in this very order of A and B)** if the number of culumns in A (called the pre-factor) is equal to the number of rows in B (called the post-factor).

Thus, if the orders of A and B are $m \times n$ and $p \times q$ respectively, then

(i) AB is defined if number of columns in A = number of rows in B, i.e. if $n=p$.

(ii) BA is defined if number of columns in B = number of rows in A, i.e., if $q=m$.

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ be two matrices conformable for the product AB, then AB is defined as the matrix $C = [c_{ij}]_{m \times p}$,

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots\ldots\ldots\ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ij}b_{kj}$

i.e. (i,j)th element of AB = sum of the products of the elements of ith row of A with the corresponding elements of jth column of B.

The rule for multiplication of two conformable matrices is called **row-by-column method.**

Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & b_{32} & c_{33} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$

Orders of A and B are 3×3 and 3×2 respectively. AB is defined and is of order 3×2

$$AB = C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}$$

where c_{11} = sum of products of elements of 1st row of A and 1st column of B.

$$= [a_{11}\ a_{12}\ a_{13}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$$

c_{12} = sum of products of elements of 1st row of A and 2nd column of B

$$= [a_{11}\ a_{12}\ a_{13}] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$$

c_{21} = sum of products of elements of 2nd row of A and 1st column of B

$$= [a_{21}\ a_{22}\ a_{23}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$$

c_{22} = sum of products of elements of 2nd row of A and 2nd column of B

$$= [a_{21}\ a_{22}\ a_{23}] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$$

c_{31} = sum of products of elements of 3rd row of A and 1st column of B

$$= [a_{31}\ a_{32}\ a_{33}] \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}$$

c_{32} = sum of products of elements of 3rd row of A and 2nd column of B

$$= [a_{31}\ a_{32}\ a_{33}] \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}$$

Thus AB =

$$\begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{11}+a_{22}b_{22}+a_{23}b_{32} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} \end{bmatrix}$$

Note : Another useful notation to remember matrix multiplication.

$$A = \begin{bmatrix} a_{1T} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & b_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix}$$

, where R_1 = 1st Row, R_2 = 2nd Row, R_3 = 3rd Row

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = [C_1\ C_2],$$

where C_1 = 1st column, C_2 = 2nd column

$$AB = \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} [C_1 C_2] = \begin{bmatrix} R_1C_1 & R_1C_2 \\ R_2C_1 & R_2C_2 \\ R_3C_1 & R_3C_2 \end{bmatrix}$$

Example. 2 If $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$,

form the products AB and BA and show that $AB \neq BA$

Solution. $AB = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$

$$= \begin{bmatrix} 1(1)-2(0)+3(1) & 1(0)-2(1)+3(2) & 1(2)-2(2)+3(0) \\ 2(1)+3(0)-1(1) & 2(0)+3(1)-1(2) & 2(2)+3(2)-1(0) \\ -3(1)+1(0)+2(1) & -3(0)+1(1)+2(2) & -3(2)+1(2)+2(0) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+0(2)+2(-3) & 1(-2)+0(3)+2(1) & 1(3)+0(-1)+2(2) \\ 0(1)+1(2)+2(-3) & 0(-2)+1(3)+2(1) & 0(3)+1(-1)+2(2) \\ 1(1)+2(2)+0(-3) & 1(-2)+2(3)+0(1) & 1(3)+2(-1)+0(2) \end{bmatrix}$$

$$= \begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

orders of AB and BA are the same but their corresponding elements are not equal.

Hence $AB \neq BA$

Example. 3 Multiplication of matrices is not commutative in general, i.e. $AB \neq BA$.

Solution. **Case I. AB is defined, but BA is not defined.**

Let A be matrix of order 2×3, and B be a matrix of 3×4.

Then AB is defined and is a matrix of order 2×4, whereas BA is not defined.

Case II. AB and BA are both defined but their orders are different.

Let A be a matrix of order 2×3 and B be a matrix of order 3×2.

The product AB is defined and is a matrix of order 2×2.

The product of BA is defined and is a matrix of order 3×3.

Since orders of AB and BA are different, $AB \neq BA$.

Case III. AB and BA are both defined and are matrices of the same order, yet AB ≠ BA (since corresponding elements are not equal).

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}$

Here AB and BA are both defined and are given by

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 22 & 3 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 4 & 8 \end{bmatrix}$$

Clearly, $AB \neq BA$.

Hence multiplication of matrices is not commutative in general, i.e. in general $AB \neq BA$.

2.3.5 PROPERTIES OF MATRIX MULTIPLICATION

(i) Matrix multiplication is not commutative in general i.e., $AB \neq BA$.

(ii) Matrix multiplication is associative i.e. $(AB)C = A(BC)$

(iii) Matrix multiplication is distributive with respect to matrix addition. i.e. $A(B+C) = AB + AC$.

(iv) If A and I are square matrices of the same order, then $AI = IA = A$.

(v) If A is a square matrix of order n, then $A \times A = A^2$, $A \times A \times A = A^3$, etc.

Also, we define $A^0 = I$

(vi) For any positive integer n, $I^n = I$.

Example. 4 Evaluate $A^2-3A+9I$, If I is the unit matrix of order 3 and $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

Solution $A^2 = A \times A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 1(1)-2(2)+3(-3) & 1(-2)-2(3)+3(1) & 1(3)-2(-1)+3(2) \\ 2(1)+3(2)-1(-3) & 2(-2)+3(3)-1(1) & 2(3)+3(-1)-1(2) \\ -3(1)+1(2)+2(-3) & -3(-2)+1(3)+2(1) & -3(3)+1(-1)+2(2) \end{bmatrix}$$

$$\therefore \quad A^2 = \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix}$$

$$\therefore \quad A^2-3A+9I_3$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} -3 \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix} +9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -5 & 11 \\ 11 & 4 & 1 \\ -7 & 11 & -6 \end{bmatrix} - \begin{bmatrix} 3 & -6 & 9 \\ 6 & 9 & -3 \\ -9 & 3 & 6 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -12-3+9 & -5-(-6)+0 & 11-9+0 \\ 11-6+0 & 4-9+9 & 1-(-3)+0 \\ -7-(-9)+0 & 11-3+0 & -6-6+9 \end{bmatrix} = \begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$$

2.3.6 Mathematical Induction

Mathematical induction is a very useful device for proving results for all positive integers. If the result to be proved involves n, where n is a positive integer, then the proof by mathematical induction consists of the following two steps :

Step 1. Verify the result for n = 1.

Step 2. Assume the result to be true for n = k and **then prove** that it is true for n = k+1.

Now the result is true for n = 1

Using step 2, the result is true for n = 1+1 = 2

⇒ The result is true for n = 2+1 = 3

⇒ The result is true for n = 3+1 = 4 and so on.

Hence the result is true for all positive integers n.

Example. 5 By Mathematical Induction, prove that if

$A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$ where n is any positive integer.

Solution We prove the result by mathematical induction

When n = 1, $A^n = \begin{bmatrix} 1+2n & -4n \\ n & 1-2n \end{bmatrix}$

$\Rightarrow$ $A^1 = \begin{bmatrix} 1+2.1 & -4.1 \\ 1 & 1-2.1 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} = A$

$\Rightarrow$ This result is true when n = 1.

Let us assume that the result is true for any positive integer k.

i.e. let $A^k = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix}$ ....(1)

Now $A^{k+1} = A^k.A = \begin{bmatrix} 1+2k & -4k \\ k & 1-2k \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$ [using (1)]

$= \begin{bmatrix} 3(1+2k)-4k & -4(1+2k)+4k \\ 3k+1-2k & -4k-1+2k \end{bmatrix} = \begin{bmatrix} 3+2k & -4-4k \\ 1+k & -1-2k \end{bmatrix}$

$= \begin{bmatrix} 1+2(k+1) & -4(k+1) \\ k+1 & 1-2(k+1) \end{bmatrix}$

$\Rightarrow$ The result is true for n = k+1

Hence, by mathematical induction, the resultr is true for all positive integers n.

2.4 TRANSPOSE OF A MATRIX :

Given a matrix A, then the matrix obtained from A by changing its rows into columns and columns into rows is called the **transpose** of A and is denoted by A' or A^t or A^c.

For example, if $A = \begin{bmatrix} 1 & 0 & 2 & 5 \\ 2 & -1 & 3 & 7 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 2 & 3 \\ 5 & 7 \end{bmatrix}$

Clearly (i) if the order of A is m × n then order of A' is n × m.

(ii) (i, j)th element of A' = (j, i)th element of A.

2.4.1 PROPERTIES OF TRANSPOSE OF A MATRIX

If A' and B' denote the transpose of A and B respectively, then

(a) **(A')' = A,** i.e., **the transpose of the transpose of a matrix is the matrix itself.**

Let $A = [a_{ij}]_{m\times n}$ be any matrix,

Order of A' is n×m. $\therefore$ Order of (A')' is m×n

$\therefore$ The matrices (A')' and A are of the same order.

Also, (i,j)th element of (A')' = (j,i)th element A'=(i,j)th element of A.

Since (A')' and A are of the same order and their corresponding elements are equal

$\therefore$ (A')' = A

(b) **(A+B)' = A'+B', i.e., the transpose of the sum of two matrices is equal to the sum of their transposes.**

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two matrices.

Order of A+B is m×n. $\therefore$ Order of (A+B)' is n × m.

A' and B' are both n×m matrices. $\therefore$ Order of A' + B' is n × m.

Thus (A+B)' and A' + B' are of the same order.

Also (i,j)th element of (A+B)' = (j, i)th element of A+B

= (j, i)th element of A + (j, i)th element of B

= (i, j)th element of A' + (i, j)th element of B'

= (i, j)th element of A' + B'

Since (A+B)' and A' + B' are of the same order and their corresponding elements are equal. $\therefore$ (A+B)' = A' + B'

(c) **(AB)' = B'A',** i.e., **the transpose of the product of two matrices is equal to the product of their transposes taken in the reverse order.**

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices conformable for the product AB.

Order of AB is m×p ∴ Order of (AB)' is p×m.

Order of B' is p×n and order of A' is n×m.

∴ Order of B'A' is p×m.

Thus, (AB)' and B'A' are of the same order.

Also, (i, j)th element of (AB)'

=(j, i)th element of AB

= sum of the products of the elements of jth row of A with the corresponding elements of ith column of B

= sum of the products of elements of ith column of B with the corresponding elements of the jth row of A

= sum of the products of elements of ith row of B' with the corresponding elements of the jth column of A'

= (i, j)th element of B'A'.

Since (AB)' and B'A' are of the same order and their corresponding elements are equal ∴ (AB)' = B'A'

2.5 SOME MORE DEFINITIONS :

2.5.1 SYMMETRIC MATRIX

A square matrix $A = [a_{ij}]$ is said to be symmetric if **A' = A** i.e. if the transpose of the matrix is equal to matrix itself.

Thus, for a symmetric matrix $A = [a_{ij}]$, $a_{ij}=a_{ji}$

i.e. (i, j)th element of A = (j, i)th element of A

For example, $\begin{bmatrix} -1 & 3 & 2 \\ 3 & 5 & 4 \\ 2 & 4 & 6 \end{bmatrix}$ $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ are symmetric matrices.

2.5.2 SKEW-SYMMETRIC MATRIX (or Anti-symmetric Matrix)

A square matrix $A = [a_{ij}]$ is said to be skew-symmetric if **A' = -A** i.e. if the transpose of the matrix is equal to the negative of the matrix.

Thus, for a skew-symmetric matrix $A = [a_{ij}]$, $a_{ij} = -a_{ji}$

i.e. (i, j)th element of A = negative of (j, i)th element of A

Putting j = i, $a_{ii} = -a_{ii}$ $\Rightarrow$ $2a_{ii} = 0$ or $a_{ii} = 0$ for all i.

Thus, all diagonal elements of a skey-symmetric matrix are zero.

For example, $\begin{bmatrix} 0 & 3 & -2 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{bmatrix}$ are skew-symmetric matrices

2.5.3 ORTHOGONAL MATRIX

A square matrix A is called an orthogonal matrix if AA' = I.

Note. AA' = I $\Rightarrow$ A'A = I.

Example. 6 If A and B are symmetric matrices, prove that AB-BA is a skew symmetric matrix.

Solution. A and B are symmetric matrices

$\Rightarrow$ A' = A and B' = B .... (1)

Now (AB - BA)' = (AB)' - (BA)' [Œ (A-B)' = A'-B']

= B'A' - A'B' [Œ (AB)' = B' A']

= BA - AB [using (1)]

= -(AB-BA)

$\therefore$ (AB - BA) is a skew-symmetric matrix.

2.6 ADJOINT OF A SQUARE MATRIX :

The adjoint of a square matrix is the transpose of the matrix obtained by replacing each element of A by its co-factor in |A|.

Adjoint of A is briefly written as **adj A**

Thus, if $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then adj A = transpose of $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

where the capital letters denote the co-factors of corresponding small letters in

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Result : If A is n-rowed square matrix, then A(adj A) = (adj A) A = $|A| \, I_n$.

Example. 7 If $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$, verify that A(adj A) = (adj A)A = |A| I.

Solution. $A = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$A_1 = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1, \quad A_2 = -\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3 \quad A_3 = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$

$B_1 = -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} = -7, \quad B_2 = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3 \quad B_3 = -\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5$

$C_1 = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5, \quad C_2 = -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = -3 \quad C_3 = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1$

$|A| = a_1A_1 + b_1B_1 + c_1C_1 = 2(-1) + 1(-7) + 3(5) = 6$

$$\text{adj } A = \text{transpose of } \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$A(\text{adj } A) = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2(-1)+1(-7)+3(5) & 2(3)+1(3)+3(-3) & 2(-1)+1(5)+3(-1) \\ 3(-1)+1(-7)+2(5) & 3(3)+1(3)+2(-3) & 3(-1)+1(5)+2(-1) \\ 1(-1)+2(-7)+3(5) & 1(3)+2(3)+3(-3) & 1(-1)+2(5)+3(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I$$

$$(\text{adj } A)\, A = \begin{bmatrix} -1 & 3 & -1 \\ -7 & 3 & 5 \\ 5 & -3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1(2)+3(3)-1(1) & -1(1)+3(1)-1(2) & -1(3)+3(2)-1(3) \\ -7(2)+3(3)+5(1) & -7(1)+3(1)+5(2) & -7(3)+3(2)+5(3) \\ 5(2)-3(3)-1(1) & 5(1)-3(1)-1(2) & 5(3)-3(2)-1(3) \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = |A|\, I$$

Hence A(adjA) = (adj A)A = |A| I.

Def : SINGULAR AND NON-SINGULAR MATRICES

A square martix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

2.7 INVERSE (OR RECIPROCAL) OF A SQUARE MATRIX :

Let A be an n-rowed square matrix. If there exists an n-rowed square matrix B such that AB = BA = I

then the matrix **A is said to be invertible** and B is called the **inverse** (or **reciprocal**) of A.

Note 1. Only square matrices can have inverse.

Note 2. Inverse of A is denoted by A^{-1}, thus $B = A^{-1}$ and $AA^{-1} = A^{-1} A = I$.

THE INVERSE OF A SQUARE MATRIX, IF IT EXISTS, IS UNIQUE

Let A be an invertible square matrix. If possible, let B and C be two inverses of A.

Then $\left.\begin{array}{l} AB = BA = I \\ AC = CA = I \end{array}\right\}$ (by def. of inverse)

Now $B = BI = B(AC)$

$= (BA)C$ [$\because$ matrix multiplication is associative]

$= IC = C$

Hence the inverse of A is unique.

Theorem. The necessary and sufficient condition for a square matrix A to possess inverse is that $|A| \neq 0$ (i.e. A is non-singular)

Proof. (a) **The condition is necessary.**

i.e. Given that A has inverse, to show that $|A| \neq 0$

Let B be the inverse of A, then $AB=BA=I \Rightarrow |AB| = |BA| = |I|$

$\Rightarrow |A|\ |B| = |B|\ |A| = I \Rightarrow |A| \neq 0$

(b) **The condition is sufficient.**

i.e. Given that $|A| \neq 0$, to show that A has inverse.

because $|A| \neq 0$

$\therefore$ Consider $B = \dfrac{\text{adj } A}{|A|}$

$$AB = A\left(\frac{\text{adj } A}{|A|}\right) = \frac{1}{|A|}(A(\text{adj } A)) = \frac{1}{|A|}(|A|\ I)$$

$= I$ [$\because(\text{adj}A)A = |A|\ I$]

$$BA = \left(\frac{\text{adj } A}{|A|}\right)A = \frac{1}{|A|}(A(\text{adj } A)) = \frac{1}{|A|}(|A|\ I)$$

$= I$ [$\because A(\text{adj } A) = |A|\ I$]

Thus $AB = BA = I$

therefore, The inverse of A exists and $A^{-1} = B$

Note. From the above theorm, we conclude that A has inverse if and only if $|A| \neq 0$

and then $A^{-1} = \dfrac{\text{adj } A}{|A|}$

Example. 8 If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, (i) find A^{-1} (ii) show that $A^3 = A^{-1}$

Sol. (i) $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix} \text{ operating } C_2+C_3$$

$$= \begin{vmatrix} 3 & 1 & 4 \\ 2 & 1 & 4 \\ 0 & 0 & 1 \end{vmatrix} \text{ expanding by third column}$$

$$= \begin{vmatrix} 3 & 1 \\ 2 & 1 \end{vmatrix} = 3-2 = 1$$

because $|A| = 1 \neq 0$ $\therefore$ A^{-1} exists.

$A_1 = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1,$ $A_2 = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1$ $A_3 = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0$

$B_1 = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2,$ $B_2 = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3$ $B_3 = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$

$C_1 = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2,$ $C_2 = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix} = 3$ $C_3 = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3$

$\therefore$ adj A = transpose of $\begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

$A^{-1} = \dfrac{1}{|A|} \text{adj } A = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$ ($|A| = 1$)

(ii) $A^2 = A \times A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3(3)-3(2)+4(0) & 3(-3)-3(-3)+4(-1) & 3(4)-3(4)+4(1) \\ 2(3)-3(2)+4(0) & 2(-3)-3(-3)+4(-1) & 2(4)-3(4)+4(1) \\ 0(3)-1(2)+1(0) & 0(-3)-1(-3)+1(-1) & 0(4)-1(4)+1(1) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -1 & 2 & -3 \end{bmatrix}$$

$$A^4 = A^2 \times A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(3)-4(0)+4(-2) & 3(-4)-4(-1)+4(2) & 3(4)-4(0)+4(-3) \\ 0(3)-1(0)+0(-2) & 0(-4)-1(-1)+0(2) & 0(4)-1(0)+0(-3) \\ -2(3)+2(0)-3(-2) & -2(-4)+2(-1)-3(2) & -2(4)+2(0)-3(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Now $A^4 = I \Rightarrow A.A^3 = A^3.A = I$

$\Rightarrow$ A^3 is the inverse of A i.e. $A^3 = A^{-1}$

Note. 1 If A is invertible, then so is A^{-1} and $(A^{-1})^{-1} = A$

A is invertible $\Rightarrow$ A^{-1} exists and $AA^{-1} = A^{-1} A = I$

$\therefore$ By definition of inverse of a matrix, A^{-1} is invertible and inverse of A^{-1} is A.

i.e. $(A^{-1})^{-1} = A$.

Note. 2 If A and B be two non-singular square matrices of the same order, then $(AB)^{-1} = B^{-1} A^{-1}$

Note. 3 If A is a non-singular square matrix, then so in A' and $(A')^{-1} = (A^{-1})'$

2.8 SOLUTION OF SIMULTANEOUS LINEAR EQUATIONS USING MATRICES:

Consider the system of equations
$$\left.\begin{matrix} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{matrix}\right] \text{(3 equations in 3 unknowns)}$$

In matrix notation, these equations can be written as

$$\begin{bmatrix} a_1x+b_1y+c_1z \\ a_2x+b_2y+c_2z \\ a_3x+b_3y+c_3z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

or $$AX = B$$

where, $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is called the co-efficient matrix,

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the column matrix of unknowns.

$B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ is the column matrix of constants.

If $d_1 = d_2 = d_3 = 0$, then B = 0 and the matrix equation AX=B reduces to AX=0. Such a system of equations is called a system of **homogenous linear equations.** If at least one of d_1, d_2, d_3 is non-zero, then $B \neq 0$

Such a system of equations is called a system of **non-homogenous linear equations.**

Solving the matrix equation AX = B means finding

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The matrix equation AX = B need not always have a solution. It may have no solution or a unique solution or an infinite number of solutions.

A system of equations having **no solution** is called an **inconsistent** system of equations.

A system of equations having **one or more solution** is called **a consistent** system of equations.

The matrix [A : B] in which the elements of A and B are written by side is called the **augmented matrix.**

For a system of homogenous linear equations AX = 0

(i) X = 0 is always a solution. This solution in which each unknown has the value zero is called the **Null Solution** or the **Trivial Solution.** Thus a homogenous system is always consistent.

A system of homogenous linear equations has either the trivial solution or an infinite number of solutions.

If A is a non-singular matrix, then the matrix equation AX = B has a unique solution

The given equation is $\quad AX = B \qquad$ ....(1)

Œ A is a non-singular matrix, $\therefore$ A^{-1} exists.

Pre-multiplying both sides of (1) by A^{-1}, we get

$A^{-1}AX = A^{-1}B \quad$ or $\quad (A^{-1}A)X = A^{-1}B$

or $\quad IX = A^{-1}B \quad$ or $\quad X = A^{-1}B$

which is the required unique solution (since A^{-1} is unique).

Example 9 Solve the system of equations :

$$5x + 3y + 7z = 4, \quad 3x + 26y + 2z = 9, \quad 7x + 2y + 11z = 5$$

with the help of matrix inversion ......(1)

Solution. $A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

Let $\quad A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 11 \end{vmatrix} = 5\begin{vmatrix} 26 & 2 \\ 2 & 11 \end{vmatrix} -3\begin{vmatrix} 3 & 2 \\ 7 & 11 \end{vmatrix} +7\begin{vmatrix} 3 & 26 \\ 7 & 2 \end{vmatrix}$$

$$= 5(286-4) -3(33-14) +7(6-182) = 1410-57-1232 = 121 \neq 0$$

$\Rightarrow$ A is non-singular $\quad\therefore A^{-1}$ exists and the unique solution of (1) is

$$X = A^{-1}B \qquad .....(2)$$

$$\text{adj } A = \begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{121}\begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

From (2), $\quad X=A^{-1}B = \frac{1}{121}\begin{bmatrix} 282 & -19 & -176 \\ -19 & 6 & 11 \\ -176 & 11 & 121 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{121}\begin{bmatrix} 77 \\ 33 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{77}{121} \\ \frac{33}{121} \\ \frac{0}{121} \end{bmatrix}$$

Hence $\quad x = \frac{7}{11}, \qquad y = \frac{3}{11}, \qquad z = 0.$

Example 10.

If $A = \begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix}$ & $B = \begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & 5 & -3 \end{bmatrix}$, find 3A-4B

Solution

$$3A-4B = 3\begin{bmatrix} 3 & 2 & 0 \\ 4 & 1 & -1 \\ 1 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 5 & 1 & 3 \\ 2 & 1 & 1 \\ -1 & 5 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & 0 \\ 12 & 3 & -3 \\ 3 & 6 & 6 \end{bmatrix} - \begin{bmatrix} 20 & 4 & 12 \\ 8 & 4 & 4 \\ -4 & 20 & -12 \end{bmatrix}$$

$$= \begin{bmatrix} -11 & 2 & -12 \\ 4 & -1 & -7 \\ 7 & -14 & 18 \end{bmatrix}$$

Example 11. If $2x+3y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$ and $3x+2y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$,

find x and y.

Solution. we are given $2x+3y = \begin{bmatrix} 2 & 2 \\ 4 & 0 \end{bmatrix}$ ---------------(1)

$$3x+2y = \begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix} \quad \text{---------------(2)}$$

Multiplying (1) by 2 & (2) by 3 & subtracting, we

$$(4x+6y)-(ax+6y) = 2\begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - 3\begin{bmatrix} -2 & 2 \\ 1 & -5 \end{bmatrix}$$

$$\Rightarrow -5x = \begin{bmatrix} 10 & 0 \\ 5 & 15 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix}$$

Putting this value in (1), we get

$$2\begin{bmatrix} -2 & 0 \\ -1 & -3 \end{bmatrix} + 3y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix}$$

$$\Rightarrow 3y = \begin{bmatrix} 2 & 3 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} -4 & 0 \\ -2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 3 \\ 6 & 6 \end{bmatrix}$$

$$\Rightarrow y = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}$$

Example 12. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ and verify that $A.A^{-1} = A^{-1}. A = I$.

Solution. Here $\quad$ **A =** $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

$$\therefore \qquad |A| = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= 1\,(3-0) - 2\,(-1-0) - 2\,(2-0) = 3+2-4 = 1 \neq 0$$

$\therefore$ A is non-singular and hence $\mathbf{A}^{-1}$ exists.

To find adj, **A**. The co-factors of **A** are

$$A_1 = (-1)^{1+1} \begin{bmatrix} 3 & 0 \\ -2 & 1 \end{bmatrix} = (3-0) = 3;$$

$$B_1 = (-1)^{1+2} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -(-1-0) = 1;$$

$$C_1 = (-1)^{1+3} \begin{bmatrix} -1 & 3 \\ 0 & -2 \end{bmatrix} = (2-0) = 2;$$

$$A_2 = (-1)^{2+1} \begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix} = -(2-4) = 2;$$

$$B_2 = (-1)^{2+2} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = (1-0) = 1;$$

$$C_2 = (-1)^{2+3} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} = -(-2-0) = 2;$$

$$A_3 = (-1)^{3+1} \begin{bmatrix} 2 & -2 \\ 3 & 0 \end{bmatrix} = (0+6) = 6;$$

$$B_3 = (-1)^{3+2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = -(0+2) = 2;$$

$$B_3 = (-1)^{3+2} \begin{bmatrix} 1 & -2 \\ -1 & 0 \end{bmatrix} = -(0+2) = 2;$$

$$C_3 = (-1)^{3+3} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = (3+2) = 5.$$

$$\text{adj. } A = [A_{ij}] = \begin{bmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{bmatrix}^t = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

Now $A^{-1} = \frac{1}{|A|}(\text{adj.}\mathbf{A}) = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$ $[\because |A| = 1]$

Verification :

$$AxA^{-1} = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} x \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3+2-4 & 2+2-4 & 6+4-10 \\ -3+3+0 & -2+3+0 & -6+6+0 \\ 0-2+2 & 0-2+2 & 0-4+5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \qquad ...(1)$$

$$A^{-1} x A = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix} x \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 -2+0 & 6+6-12 & -6+0+6 \\ 1 -1+0 & 2+3-4 & -2+0+2 \\ 2 -2+0 & 4+6-10 & -4+0+5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I} \qquad ...(2)$$

From (1) and (2), we have $A.A^{-1} = A^{-1}A=I$.

Example 13. Solve the following system of equations :

$$x + 2y = 4$$

$$2x+5y = 9.$$

Solution. The given system of equations can be expressed in the form of matrix equation **AX=B**

where $A= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$; $\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}$; $\mathbf{B} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = 1.5-2.2=5-4=1 \neq 0$$

$\therefore$ A is non-singular and hence the system has a unique solution

$A_1 = 5$; $B_1 = -2$ $A_2 = -2$; $B_2 = 1$

adj. A = $[A_{ij}] = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|\mathbf{A}|}(\text{adj.}\mathbf{A}) = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

Unique Solution of (1) is **X=A-[1]B**

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 5.4 \quad 2.9 \\ -2.4+1.9 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Hence **x=2, y =1.**

Example 14. *Solve the following system of equations :*

$$2x+8y+5z=5$$

$$x+y+z=-2$$

$$x+2y-z=2.$$

Solution. The given equations can be expressed in the form of matrix equation **AX = B**

where $A = \begin{bmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$

$$|A| = \begin{bmatrix} 2 & 8 & 5 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= 2(-1-2) -8(-1-1) + 5(2-1)$$

$$= -6+16+5=15 \neq 0$$

$\therefore$ A is non singular and hence the system has a unique solution given by

$$\mathbf{X=A^{-1}B.}$$

$$\text{adj. } \mathbf{A} = [A_{ij}] = \begin{bmatrix} -3 & 2 & 1 \\ 18 & -7 & 4 \\ 3 & 3 & -6 \end{bmatrix}^t = \begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|\mathbf{A}|}(\text{adj.}\mathbf{A}) = \frac{1}{15} \begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}$$

Therefore, $\quad$ **X=A-¹B.**

i.e
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{15}\begin{bmatrix} -3 & 18 & 3 \\ 2 & -7 & 3 \\ 1 & 4 & -6 \end{bmatrix}\begin{bmatrix} 5 \\ -2 \\ 2 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} -15-36+6 \\ 10+14+6 \\ 5-8-12 \end{bmatrix}$$

$$= \frac{1}{15}\begin{bmatrix} -45 \\ 30 \\ -15 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \\ -1 \end{bmatrix}$$

Hence **x = - 3, y = 2, z = -1.**

2.9 SELF ASSESSMENT QUESTIONS :

1. If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & -1 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 0 & 2 \end{bmatrix}$

find 3A+4B.

2. If $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{bmatrix}$, Verify that

2 (A+B) = 2A+2B.

3. If $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$ & $C \begin{bmatrix} 1 & 0 & 1 \\ 2 & 3 & 0 \end{bmatrix}$

Verify that A (B+C) = AB+AC

4. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, find f (A), where $f(x) = x^2-5x+7$

5. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, Evaluate $A^3-23A-40\,I$

6. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 1 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ & $B = \begin{bmatrix} 3 & 0 & 4 \\ 1 & 2 & 5 \\ 2 & 2 & 4 \end{bmatrix}$

Check whether (AB)' = B'A'

7. Find the adjoint of matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$$

and verify that A (adjA) = |A | I

8. Find inverse of the matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$$

9. If $A = \begin{bmatrix} 3 & 5 \\ 2 & 7 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ Verify that $(AB)^{-1} = B^{-1}A^{-1}$,

10. Solve the system of liner equations.

x+2y-3z=-4, 2x+3y+2z = 2, 3x-3y-4z =11

11. Solve the system of liner equations by matrix method

2x+3y+3z = 5, x-2y+z=-4, 3x-y-2z=3

2.10 KEY WORDS :

Matrix, transpose, adjoint, inverse of matrix.

2.11 SUGGESTED READINGS :

1. Grewal, B.S. - Engineering Mathematics.
2. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.
3. Ramana, B.V. - Higher Engineering Mathematics.
4. Aggarwal, R.S. - Modern Approach of Mathematics.

In the adjoining figure, the initial line OX moves anti-clockwise to the terminal line OP from the common point O, called the vertex, to trace a positive angle.

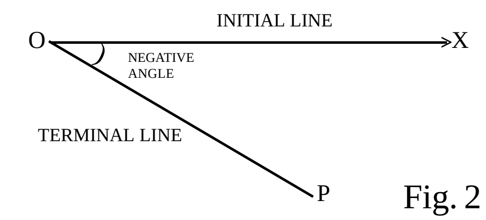


Fig. 2

A similar movement clockwise will trace a negative angle.

3.2.2 QUADRANTS

Let the two perpendiculars line $X'OX$ and $Y'OY$ divide the plane into four parts, each one of them being called a *quadrant*. Conventionally the region XOY is called the *First quadrant*, the region YOX' is called the *second quadrant*, the region $X'OY'$ is called the *Third quadrant* and the region $Y'OX$ is called the *Fourth quadrant*.

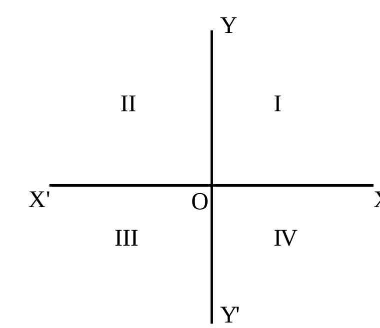


Fig. 3

3.3 MEASUREMENT OF ANGLES :

In geometry, an angle is generally measurement in terms of a right angle. In trignometry there are three systems for the measurement of angles.

3.3.1 Sexagesimal System (or the English System). In this system a right angle is divided into small parts as shown below :

1 rt. angle = 90 degrees (written as 90°)

1 degree = 60 minutes (written as $60'$)

1 minute = 60 seconds (written as $60''$)

3.3.2 The Centesimal System (or the French System). In this system, a right angle is divided and subdivided as shown below :

1 rt. angle = 100 grades (written as 100^g)

1 grade = 100 minutes (written as $100'$)

1 minutes = 100 seconds (written as $100''$)

The minutes and seconds used in the centesimal system are distinct from those used in the sexagesimal system.

A right angle connects the two systems, an angle in the first system can be converted into the units of the second system and vice versa.

one rt. angle contains 90^o and 100^g

Œ $90^1 = 100^g$ L $1^1 = \frac{10^g}{9}$

Thus to change degrees into grades, multiply by $\frac{10}{9}$, e.g.

$$63^1 = 63 \times \frac{10}{9} = 70^g$$

Similarly to change grades into degrees multiply by $\frac{9}{10}$, e.g.

$$40^g = 40 \times \frac{9}{10} = 36^1 .$$

3.3.3 The Circular System (or the Circular Measure)

This system is commonly used for measurement of angles. In this system, the unit of measurement is a radian.

The radian is defined as *an angle subtended at the centre of a circle by an arc equal in length to the radius of the circle.*

Let us draw a circle around the center *O* with any radius. From any point *A* on the circumference cut off an arc *AB* = radius of the circle. Join *OA* and *OB*. Then

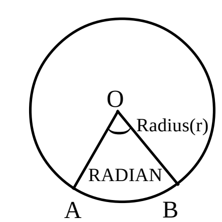
$\angle AOB$ = 1 radian, circular measure of an angle is the number of radians it contains.

Since, the angles at the centre of a circle are proportional to the arcs subtended through them, we have

$$\frac{\angle AOB}{\text{arc } AB} = \frac{2\pi \text{ (i.e., total angle at O)}}{\text{Circumference}}$$

$$\frac{\angle AOB}{\text{Total angle at } O} = \frac{\text{arc } AB}{\text{Circumference}} = \frac{r}{2\pi r} = \frac{1}{2\pi}$$

$$\Rightarrow \quad \frac{1 \text{ radian}}{4 \text{ rt. } \angle s} = \frac{1}{2\pi}$$



$$\Rightarrow \quad 1 \text{ radian} = \frac{4 \text{ rt. } \angle s}{2\pi} = \frac{2}{\pi} \text{ rt. –s.}$$

$$\Rightarrow \quad \pi \text{ radians} = 2 \text{ rt. } \angle s = 180^{1} = 200^{g}$$

which is the relation between three systems for the measurement of angles. Through this, given any one system we can derive in any other system.

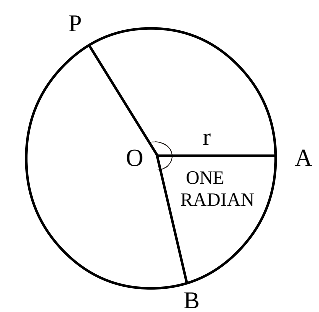
3.4 AN IMPORTANT FORMULA :

To prove that the number of radians in an angle subtended by an arc of a circle at the centre = $\mathbf{\dfrac{arc}{radius}}$ **.**

Let *r* be the radius of the circle with centre *O*. Let *AOP* be the angle subtended by an arc *AP* of a circle at its centre.

Cut off arc $AB = r$, and join *OB*, then $\angle AOB = 1$ radian [Def.].

Since $\angle$s at the centre of a circle are propotional to the arcs on which they stand



$$\therefore \quad \frac{\angle AOP}{\angle AOB} = \frac{\text{arc } AP}{\text{arc } AB} = \frac{\text{arc } AP}{r}$$

$$\angle AOP = \frac{\text{arc } AP}{r} \angle AOB = \frac{\text{arc } AP}{r} \times 1 \text{ Radian}$$

$$= \frac{\text{arc } AP}{\text{radius}} \text{(in Radians).}$$

Note. If θ is the number of radians in the angle subtended at the centre of a circle by an arc whose length is l and r is the radius of the circle, then $\theta = \frac{l}{r}$.

Example 1. Express in radians the angle $50^{\circ}, 37', 30''$

Solution $\quad 30'' = \frac{30}{60} = \frac{1'}{2}$

$$37' 30'' = 37\frac{1'}{2} = \frac{75}{2} \times \frac{1^{\circ}}{60} = \frac{5^{\circ}}{8}$$

$$50^{\circ} 37' 30'' = 50\frac{5^{\circ}}{8} = \frac{405^{\circ}}{8} = \frac{405}{8} \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{9\pi}{32} \text{ radians.}$$

Example 2. The angles of a triangle are in A.P. and the number of degrees in the least is to the number of radians in the greatest as 45 : π ; find the angles in degrees.

Solution. Let the angles of triangle be $(a-d)^\circ$, a°, $(a+d)^\circ$

$\therefore \quad (a-d)+a+(a+d) = 180^\circ$

or $\quad 3a = 180^\circ$ or $a=60^\circ$

Now least angle $= (a-d)^\circ = (60-d)^\circ$

greatest angle $=(a+d)^\circ = (60+d)^\circ = (60+d)\frac{\pi}{180}$ radians

According to the problem, we have

$$\frac{(60-d)}{(60+d)\frac{\pi}{180}} = \frac{45}{\pi} \quad \text{or} \quad \frac{180(60-d)}{(60+d)} = 45$$

or $\quad 4(60-d) = 60+d \quad$ i.e., $\quad 5d=180^\circ \quad$ or $\quad d=36^\circ$

$\therefore \quad$ Angles are $(60-36)^\circ$, 60°, $(60+36)^\circ$

i.e., $\quad 24^\circ, 60^\circ, 96^\circ$.

ALWAYS REMEMBER

1. As 90° = 1 rt.– = $\frac{\pi}{2}$ radians, $\therefore$ it follows $1^\circ = \frac{\pi}{180}$ radians

2. 1 radian = $\frac{180^\circ}{\pi}$

Example 3. A wire in the form of a semi-circle of radius 8 cm is bent and placed along the circumference of a circular hoop; it subtends an angle of 30° at the centre of the hoop, find the radius of the hoop.

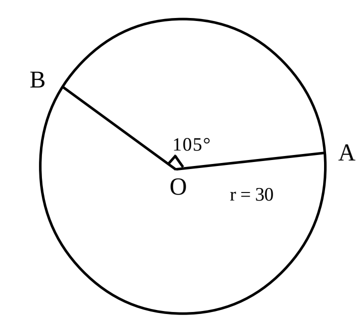
Solution. Here $l = \frac{1}{2}$ (circumference of circle of radius 8 cm)

$= \frac{1}{2} \times 2\pi .8 = 8\pi$ cm and $\frac{l}{r}$ $\theta = 30^1 = \frac{l}{\theta}$ radians

Then $\theta =$ or $\frac{1}{2\pi}$ $r = \frac{\pi}{6}$ gives

$r = 8\pi$ x $\quad = 48$ cm.

Example 4. A horse is tethered to a stake by a rope 30m long. If the horse moves along the circumference of a circle always keeping the rope tight, find the distance travelled by the horse when the rope has traced out an angle of 105°.



Solution. Let O denote the stake and OA, the initial position of the rope, Let OB be the final position of the rope.

$$\angle AOB = 105°$$

$$= 105 \text{ x } \frac{\pi}{180} \text{ radians}$$

$$= \frac{7\pi}{12} \text{ radians}$$

Let l meter be the distance covered by the horse

$$l = r\,\theta = 30 \text{ x } \frac{7\pi}{12} = 30 \text{ x } \frac{7}{12} \text{ x } \frac{22}{7}$$

$$= 55 \text{ m.}$$

Example 5. Find the length of an arc which subtends an angle 120° at the centre of a circle whose radius is 6 cm.

Solution. Here $r = 6$, $\theta = 120^\circ = 120 \times \frac{\pi}{180}$ radians

Substituting these values in the formula, we have

$$l = r\theta = 6 \times 120 \times \frac{\pi}{180} = 4\pi$$

$$= 4 \times 3.142 = 12.568 \text{ cm.}$$

Example 6. The large hand of a big clock is 3 feet long. How many inches does its extremity move in 10 minutes time?

Solution. In 60 minutes, the minute hand turns through 360°.

$\therefore$ In 10 minutes it turns through $= \frac{360 \times 10}{60} = 60^\circ$

$\therefore$ $\theta = 60 \times \frac{\pi}{180}$ radians

r = radius of the circle = 36 inches

$\therefore$ l = length of the arc = $r\theta$

$$= 36 \times 60 \times \frac{22}{7} \times \frac{1}{180}$$

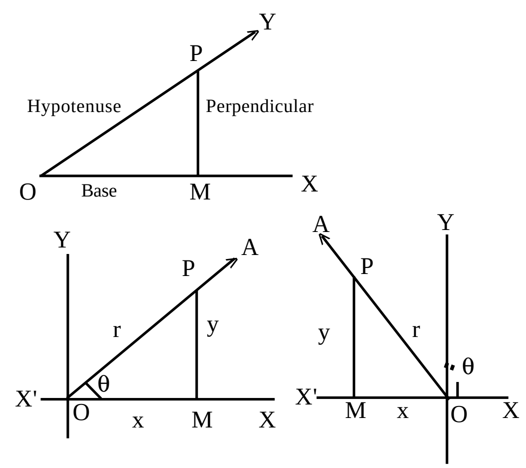
$$= \frac{264}{7} = 37.7 \text{ inches.}$$

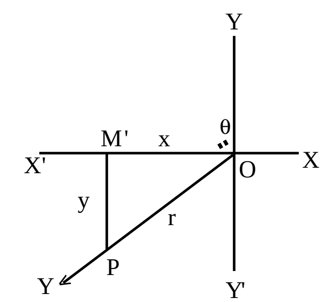
3.5 TRIGONOMETRIC FUNCTIONS :

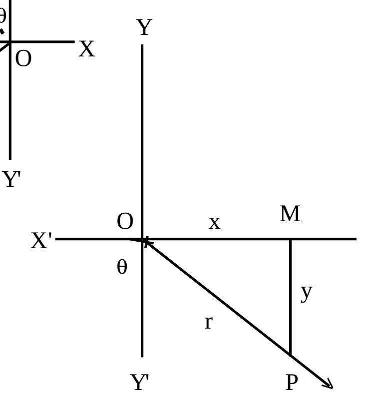
Trigonometric functions are relations between any two of the three sides of a triangle. For the sake of simplicity a right-angled triangle is taken as a starting point to explain these relations.

A Right Angle. Let *XOY* be any angle θ. Take any point *P* on *OY* and draw *PM* perpendicular to *OX*. A right angled ΔOMP is formed. If θ is taken as the angle of reference, *MP*, the side opposite to θ is called the perpendicular and *OP*, the side opposite to the right angle is called the hypotenuse and *OM* the third side is called the base.

General Angles. Let a straight line *OA*, starting form the position *OX* and rotating round *O* trace out an angle *XOA*. Let θ be the measure of the angle *XOA*. This angle can be of any magnitude. From any point *P* in the final position of the revolving line *OA*, draw *PM* perpendicular to *OX* or *XO* produced if necessary as shown below :







3.6 TRIGONOMETRY RATIOS :

Now, the three sides *OM*, *OP* and *MP* can be arranged, two at a time in six different ways and hence six ratios can be formed with them. These six ratios are called the trigonmetric functions or *t*-ratios or circular functions and are defined as follows :

1. The ratio of the perpendicular to the hypotenuse is called the sine of the angle θ and is written as $\sin\theta$.

$$\sin\theta = \frac{\text{Perp.}}{\text{Hyp.}} = \frac{MP}{OP} = \frac{y}{r} = \frac{\text{Ordinate}}{r} .$$

2. The ratio of the base to the hypotenuse is called the cosine of the angle θ and is written as cos θ.

$$\cos\theta = \frac{\text{Base}}{\text{Hyp.}} = \frac{OM}{OP} = \frac{x}{r} = \frac{\text{Abscissa}}{r} .$$

3. The ratio of the perpendicular to the base is called the tangent of the angle θ and is written as $\tan\theta$.

$$\tan\theta = \frac{\text{Perp.}}{\text{Base}} = \frac{MP}{OM} = \frac{y}{x} = \frac{\text{Ordinate}}{\text{Abscissa}}$$

The following three ratios are reciprocals of the above ratios.

4. The ratio of the hypotenuse to the perpendicular is called the cosecant of the angle θ and is written as cosecθ .

$$\text{cosec}\theta = \frac{\text{Hyp.}}{\text{Perp.}} = \frac{OP}{MP} = \frac{r}{y} = \frac{r}{\text{Ordinate}}.$$

5. The ratio of the hypotenuse to the base is called the secant of the angle θ and is written as secθ .

$$\sec\theta = \frac{\text{Hyp.}}{\text{Base}} = \frac{OP}{OM} = \frac{r}{x} = \frac{r}{\text{Abscissa}}.$$

6. The ratio of the base to the perpendicular is called the cotangent of the angle θ and is written as cotθ .

$$\cot\theta = \frac{\text{Base}}{\text{Perp.}} = \frac{OM}{MP} = \frac{x}{y} = \frac{\text{Abscissa}}{\text{Ordinate}}.$$

3.7 FUNDAMENTAL RELATIONS BETWEEN TRIGONOMETIC FUNCTIONS:

3.7.1. Reciprocal Relations. The following relations follow directly from the definitions of *t*-ratios.

(i) $\sin\theta \times \text{cosec}\theta = 1$

$\therefore \quad \sin\theta = \dfrac{1}{\text{cosec}\theta}$ and $\text{cosec}\theta = \dfrac{1}{\sin\theta}$.

(ii) $\cos\theta \times \sec\theta = 1$

$\therefore \quad \cos\theta = \dfrac{1}{\sec\theta}$ and $\sec\theta = \dfrac{1}{\cos\theta}$

(ii) $\tan\theta \times \cot\theta = 1$

$$\therefore \qquad \tan\theta = \frac{1}{\cot\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta}.$$

3.7.2 Quotient Relations :

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$.

we have $\sin\theta = \dfrac{y}{r}$, $\cos\theta = \dfrac{x}{r}$

$\therefore$ (i) $\dfrac{\sin\theta}{\cos\theta} = \dfrac{y/r}{x/r} = \dfrac{y}{x} = \tan\theta$, (ii) $\dfrac{\cos\theta}{\sin\theta} = \dfrac{x/r}{y/r} = \dfrac{x}{y} = \cot\theta$.

3.2.3 Square Relations :

(i) $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2\theta + \cos^2\theta = \frac{y^2}{r^2} + \frac{x^2}{r^2} = \frac{y^2+x^2}{r^2} = \frac{r^2}{r^2} = 1$$

From this relation, we can obtain

$\sin^2\theta = 1-\cos^2\theta$ and $\cos^2\theta = 1-\sin^2\theta$.

(ii) $\sec^2\theta - \tan^2\theta = 1$

Œ $\sec^2\theta - \tan^2\theta = \dfrac{r^2}{x^2} - \dfrac{y^2}{x^2} = \dfrac{r^2-y^2}{x^2} = \dfrac{x^2}{x^2} = 1$

It follows from above that

$\sec^2\theta = 1+\tan^2\theta$ and $\tan^2\theta = \sec^2\theta - 1$

(iii) $\text{cosec}^2\theta - \cot^2\theta = 1$

e $1+\cot^2\theta = \text{cosec}^2\theta$

Œ $\text{cosec}^2\theta - \cot^2\theta = \dfrac{r^2}{y^2} - \dfrac{x^2}{y^2} = \dfrac{r^2-x^2}{y^2} = \dfrac{y^2}{y^2} = 1$

The above three inter-relations are very important and are called Identities. If any one of the trigonometric function is given, the remaining can easily be found by using the above relations (I), (II) or (III).

Note : It should be noted that sin θ does not mean sin × θ, i.e., sin is not a multipler. The sin θ is correctly read as "sin of angle θ ". Similar is the case with other trigonometric ratios. Further $\frac{1}{\sin\theta}$

$(\sin\theta)^2$ is written as $\sin^2\theta$ (read : sine square θ)

$(\sin\theta)^3$ is written as $\sin^3\theta$ (read : sine cube θ)

Similarly, for other *t*-ratios of θ.

But $(\sin\theta)^{-1}$ is not written as $\sin^{-1}\theta$. but $(\sin\theta)^{-1}$ = .

Example 1. Prove that

$$2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1 = 0.$$

Solution. L.H.S. $= 2[(\sin^2\theta)^3 + (\cos^2\theta)^3] - 3(\sin^4\theta + \cos^4\theta) + 1$

$$= 2(\sin^2\theta + \cos^2\theta)(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$= 2.1.(\sin^4\theta + \cos^4\theta - \sin^2\theta\cos^2\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$$

$$= -(\sin^4\theta + \cos^4\theta) - 2\sin^2\theta\cos^2\theta + 1$$

$$= -[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta] - 2\sin^2\theta\cos^2\theta + 1$$

$$= -1 + 2\sin^2\theta\cos^2\theta - 2\sin^2\theta\cos^2\theta + 1$$

$$= 0 = \text{R.H.S.}$$

Example 2. Prove that

$$\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}, \ \theta \text{ is not equal to } 0.$$

Solution. L.H.S. = $\frac{\sin\theta}{1-\cos\theta} \cdot \frac{1+\cos\theta}{1+\cos\theta}$

$$= \frac{\sin\theta(1+\cos\theta)}{1-\cos^2\theta} = \frac{\sin\theta(1+\cos\theta)}{\sin^2\theta}$$

$$= \frac{1+\cos\theta}{\sin\theta} = \text{R.H.S.}$$

Example 3. Prove that

$$\sqrt{\frac{1-\sin A}{1+\sin A}} = \sec A - \tan A.$$

Solution. L.H.S. = $\sqrt{\frac{1-\sin A}{1+\sin A}}$

$$= \sqrt{\frac{1-\sin A}{1+\sin A} \times \frac{1-\sin A}{1-\sin A}}$$

$$= \frac{1-\sin A}{\sqrt{1-\sin^2 A}} = \frac{1-\sin A}{\cos A} = \frac{1}{\cos A} - \frac{\sin A}{\cos A}$$

$$= \sec A - \tan A = \text{R.H.S.}$$

Example 4. Prove that

$$(\operatorname{cosec}\theta - \sin\theta)(\sec\theta - \cos\theta)(\tan\theta + \cot\theta) = 1$$

Solution. we have to simplify the L.H.S. This simplification is done by expressing all t-ratios in terms of the sine and cosine by the use of the formula :

$$\operatorname{cosec} = \frac{1}{\sin\theta},\ \sec\theta = \frac{1}{\cos\theta},\ \tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta}$$

L.H.S. $= \left(\frac{1}{\sin\theta} - \sin\theta\right)\left(\frac{1}{\cos\theta} - \cos\theta\right)\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right)$

$$= \frac{1-\sin^2\theta}{\sin\theta} \times \frac{1-\cos^2\theta}{\cos\theta} \times \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$$

But $1-\sin^2\theta = \cos^2\theta,\ 1-\cos^2\theta = \sin^2\theta,\ \sin^2\theta + \cos^2\theta = 1$

L.H.S. = $\frac{\cos^2\theta}{\sin\theta} \times \frac{\sin^2\theta}{\cos\theta} \times \frac{1}{\sin\theta\cos\theta} = 1 = \text{R.H.S.}$

Example 5 Prove that

$$\sin A\,(1+\tan A) + \cos A\,(1+\cot A) = \sec A + \operatorname{cosec} A.$$

Solution. L.H.S. = sin A (1+tan A) + cos A (1+cot A)

$$= \sin A\left(1+\frac{\sin A}{\cos A}\right) + \cos A\left(1+\frac{\cos A}{\sin A}\right)$$

$$= \sin A\left(\frac{\sin A+\cos A}{\cos A}\right) + \cos A\left(\frac{\sin A+\cos A}{\sin A}\right)$$

$$= (\sin A+\cos A)\left(\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}\right)$$

$$= (\sin A+\cos A) \times \frac{\sin^2 A+\cos^2 A}{\cos A\sin A}$$

$$= \frac{\sin A+\cos A}{\cos A\sin A} = \frac{\sin A}{\cos A\sin A} + \frac{\cos A}{\sin A\cos A}$$

$$= \frac{1}{\cos A} + \frac{1}{\sin A} = \sec A + \operatorname{cosec} A = \text{R.H.S.}$$

Example 6. Show that

$$(1+\cot\theta - \operatorname{cosec}\theta)\,(1+\tan\theta+\sec\theta) = 2.$$

Solution. We have

$$\text{L.H.S.} = \left(1+\frac{\cos\theta}{\sin\theta} - \frac{1}{\sin\theta}\right)\left(1+\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}\right)$$

$$= \frac{\{(\sin\theta+\cos\theta)-1\}\ \{(\sin\theta+\cos\theta)+1\}}{\sin\theta\cos\theta}$$

$$= \frac{(\sin\theta+\cos\theta)^2-1}{\cos\theta\sin\theta} = \frac{\sin^2\theta+\cos^2\theta+2\sin\theta\cos\theta-1}{\cos\theta\sin\theta}$$

$$= \frac{1+2\sin\theta\cos\theta-1}{\sin\theta\cos\theta} \qquad [\because \sin^2\theta+\cos^2\theta=1]$$

$$= \frac{2\sin\theta\cos\theta}{\sin\theta\cos\theta} = 2 = \text{R.H.S.}$$

Example 7 Show that

$$\frac{\tan\theta}{\sec\theta-1} + \frac{\tan\theta}{\sec\theta+1} = 2\,\text{cosec}\theta$$

Solution. L.H.S. = $\dfrac{\tan\theta}{\sec\theta-1} + \dfrac{\tan\theta}{\sec\theta+1}$

$$= \tan\theta\left(\frac{1}{\sec\theta-1} + \frac{1}{\sec\theta+1}\right)$$

$$= \tan\theta\left[\frac{\sec\theta+1+\sec\theta-1}{(\sec^2\theta-1)}\right] \qquad [\text{Œ } \sec^2\theta = 1+\tan^2\theta]$$

$$= \tan\theta \cdot \frac{2\sec\theta}{\tan^2\theta} = 2.\frac{\sec\theta}{\tan\theta} = 2.\frac{1}{\cos\theta}\,\frac{\cos\theta}{\sin\theta}$$

$$= 2\,\text{cosec}\theta = \text{R.H.S.}$$

Example 8 Prove that

$$\frac{\tan\theta+\sec\theta-1}{\tan\theta-\sec\theta+1} = \frac{1+\sin\theta}{\cos\theta}$$

Solution. L.H.S. = $\dfrac{\tan\theta+\sec\theta-(\sec^2\theta-\tan^2\theta)}{\tan\theta-\sec\theta+1}$

$$= \frac{(\tan\theta+\sec\theta)-(\sec\theta-\tan\theta)(\sec\theta+\tan\theta)}{\tan\theta-\sec\theta+1} \qquad [\text{Œ } 1 = \sec^2\theta-\tan^2\theta]$$

$$= \frac{(\sec\theta+\tan\theta)(1-\sec\theta+\tan\theta)}{\tan\theta-\sec\theta+1}$$

$$= \sec\theta+\tan\theta = \frac{1+\sin\theta}{\cos\theta} = \text{R.H.S.}$$

Example 9. *If* $\cos\theta+\sin\theta = \sqrt{2}\cos\theta$

Prove that $\cos\theta-\sin\theta = \sqrt{2}\sin\theta$

Solution. Given $\sin\theta = \sqrt{2}\cos\theta-\cos\theta$

$\Rightarrow \quad (\sqrt{2}-1)\cos\theta = \sin\theta$

$$\Rightarrow \quad \cos\theta = \frac{\sin\theta}{(\sqrt{2}-1)} = \frac{\sin\theta}{(\sqrt{2}-1)} \times \frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}$$

$$= (\sqrt{2}+1)\sin\theta = \sqrt{2}\sin\theta + \sin\theta$$

$\Rightarrow \quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta.$

Example 10. Eliminate θ between the equations

$$a\sec\theta + b\tan\theta + c = 0$$

$$p\sec\theta + q\tan\theta + r = 0.$$

Solution. Solving the given equations for secθ and tanθ by cross-multiplication,

$$\frac{\sec\theta}{br-qc} = \frac{\tan\theta}{cp-ra} = \frac{1}{aq-pb}$$

We know that $\sec^2\theta - \tan^2\theta = 1$

$$\left(\frac{br-qc}{aq-pb}\right)^2 - \left(\frac{cp-ra}{aq-pb}\right)^2 = 1.$$

or $\quad (br-qc)^2-(cp-ra)^2=(aq-pb)^2.$

Example 11. Eliminate θ between the equations.

$$x = a\sin\theta,\ y = b\tan\theta.$$

Solution. $x = a\sin\theta$ gives $\frac{x}{a} = \sin\theta$ or $\operatorname{cosec}\theta = \frac{a}{x}$ ...(1)

$y = b\tan\theta$ gives $\frac{y}{b} = \tan\theta$ or $\cot\theta = \frac{b}{y}$ ...(2)

Square (1) and (2) and subtract, $\operatorname{cosec}^2\theta - \cot^2\theta = \frac{a^2}{x^2} - \frac{b^2}{y^2}$

$$1 = \frac{a^2}{x^2} - \frac{b^2}{y^2}.$$

Example 12. Prove that $\frac{\cos\theta}{1-\tan\theta} + \frac{\sin\theta}{1-\cot\theta} = \sin\theta + \cos\theta$

Sol. $$\text{L.H.S.} = \frac{\cos\theta}{1-\frac{\sin\theta}{\cos\theta}} + \frac{\sin\theta}{1-\frac{\cos\theta}{\sin\theta}} = \frac{\cos^2\theta}{\cos\theta-\sin\theta} + \frac{\sin^2\theta}{\sin\theta-\cos\theta}$$

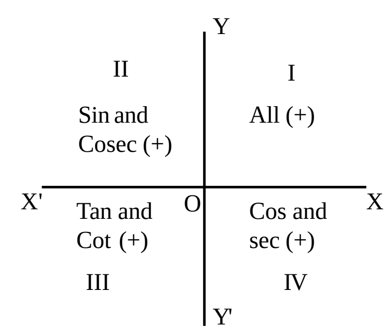
$$= \frac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta} = \cos\theta + \sin\theta = \text{R.H.S.}$$

3.8 SIGNS OF TRIGONOMETRIC RATIOS :

The radius vector is always positive in whichever quadrant it lies, therefore, the sign of a trigonometric ratio of an angle will always depend on the sign of the co-ordinates.

Now, in the first quadrant all t-functions are positive because both the co-ordinates are positive. In the second quadrant, x-coordinates is negative, therefore, all t-functions which involve x-coordinate are negative, i.e., cosθ and tanθ and their reciprocals while sinθ and cosecθ will be positive. In the third quadrant both the coordinates are negative, therefore, *t*-ratios which involve both these such as tanθ and cosθ are positive but all others are negative. In the fourth quadrant x-coordinate is positive but y-coordinate is negative therefore, cosθ and secθ which do not involve y-coordinate are positive and the rest are negative. The following figure is a good aid to memory.

To remember the signs, the phrase "After School to College" where we may take A(all), that is, S(sine), T(tangent) and C(cosine) with their reciprocals as positive in the first quadrant. Only S(sine) and its reciprocal is positive in the second quadrant, only T(tangent) and its reciprocal is positive in the third quadrant and only C(cosine) and its reciprocal is positive in the fourth quadrant.



Example 13. If θ is in the fourth quadrant and $\cos\theta = \frac{5}{13}$, find the value of

$$\frac{13\sin\theta + 5\sec\theta}{5\tan\theta + 6\,\text{cosec}\,\theta}$$

Solution. $\cos\theta = \frac{OM}{OP} = \frac{5}{13}$

∴ $OM = 5$, $OP = 13$ and

$MP^2 = OP^2 - OM^2$

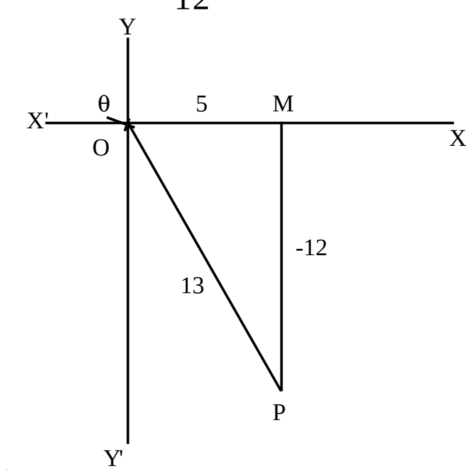
$= 13^2 - 5^2 = 144$

$MP = -12$

(MP is -ive in the 4th quadrant)

$\sin\theta = -\frac{12}{13}$, $\sec\theta = \frac{13}{5}$,

$\tan\theta = -\frac{12}{5}$ and $\operatorname{cosec}\theta = -\frac{13}{12}$.



Given expression

$$= \frac{13\sin\theta + 5\sec\theta}{5\tan\theta + 6\operatorname{cosec}\theta}$$

$$= \frac{13\left(\frac{-12}{13}\right) + 5\left(\frac{13}{5}\right)}{5\left(\frac{12}{5}\right) + 6\left(\frac{-13}{12}\right)} = \frac{-12 + 13}{-12 - \frac{13}{2}} = -\frac{2}{37}.$$

Example 14. If sinθ .secθ = -1 and θ lies in the second quadrant, find sinθ and secθ.

Solution. We are given that

$\sin\theta . \sec\theta = -1$

i.e., $\frac{\sin\theta}{\cos\theta} = -1$ or $\tan\theta = -1$.

$\therefore \quad \tan\theta = \frac{MP}{OP} = -1$

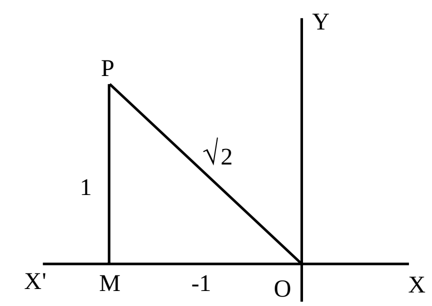
Since θ lies in the second quadrant, *MP* is +ve and *OM* is -ve.

$\therefore$ If *MP* = 1, then *OM* = -1.

Also $OP = \sqrt{MP^2+OM^2}$

$= \sqrt{1+1} = \sqrt{2}$

$\therefore \quad \sin\theta = \frac{MP}{OP} = \frac{1}{\sqrt{2}}$ and $\sec\theta = \frac{OP}{MP} = \frac{\sqrt{2}}{-1} = -\sqrt{2}$

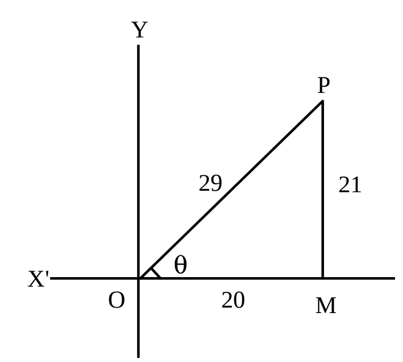


Example 15. If $\sin\theta = \frac{21}{29}$, Prove that

(i) $\sec\theta + \tan\theta = \frac{5}{2}$ **, if θ lies between 0 and** $\frac{\pi}{2}$.

(ii) What will be the value of the expression when θ lies between $\frac{\pi}{2}$ **and** π

Solution. (i) $\sin\theta = \frac{MP}{OP} = \frac{21}{29}$



$\therefore$ *MP* = 21 and *OP* = 29

Also $OM^2=OP^2-MP^2$

$= 29^2-21^2=400$

$\Rightarrow \quad OM=20$

(*OM* is +ve in the first quadrant)

$\therefore \quad \sec\theta = \dfrac{OP}{OM} = \dfrac{29}{20}$

and $\quad \tan\theta = \dfrac{MP}{OM} = \dfrac{21}{20}$

Hence $\quad \sec\theta + \tan\theta = \dfrac{29}{20} + \dfrac{21}{20} = \dfrac{5}{2}$

(ii) When θ lies between $\dfrac{\pi}{2}$ and π, i.e., in the second quadrant, *OM* is negative.

$\therefore \quad \sec\theta = \dfrac{29}{-20} = \dfrac{-29}{20}$ and $\tan\theta = \dfrac{21}{-20} = \dfrac{-21}{20}$

$\therefore \quad \sec\theta + \tan\theta = -\dfrac{29}{20} - \dfrac{21}{20} = -\dfrac{5}{2}.$

Example 16. If $\cot\theta = \dfrac{12}{5}$, $\left(\pi < \theta < \dfrac{3\pi}{2}\right)$, find the value of $\sec\theta$ and $\sin\theta$.

Solution. $\quad \text{cosec}^2\theta = 1+\cot^2\theta = 1+\dfrac{144}{25} = \dfrac{169}{25}$

$\text{cosec}\,\theta = -\dfrac{13}{5}$

(Since $\text{cosec}\,\theta$ is negative, θ being in the third quadrant)

$\sin\theta = -\dfrac{5}{13}$

Again $\quad \sec\theta = \dfrac{1}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta} \cdot \dfrac{1}{\sin\theta} = \tan\theta\,.\,\text{cosec}\,\theta$

$= \dfrac{5}{12} \cdot \dfrac{-13}{5} = -\dfrac{13}{12}.$

Example 17. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$,

Prove that $\quad \cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

Solution. $\quad \cos\theta - \sin\theta = \sqrt{2}\sin\theta$

$\cos\theta = \sqrt{2}\sin\theta + \sin\theta = (\sqrt{2}+1)\sin\theta$

$\sin\theta = \dfrac{1}{\sqrt{2}+1}\cos\theta$

$= \dfrac{1}{\sqrt{2}+1} \cdot \dfrac{\sqrt{2}-1}{\sqrt{2}-1} \cdot \cos\theta = \dfrac{\sqrt{2}-1}{(\sqrt{2})^2-(1)^2} \cdot \cos\theta$

$\sin\theta = \sqrt{2}\cos\theta - \cos\theta$

$\cos\theta + \sin\theta = \sqrt{2}\cos\theta.$

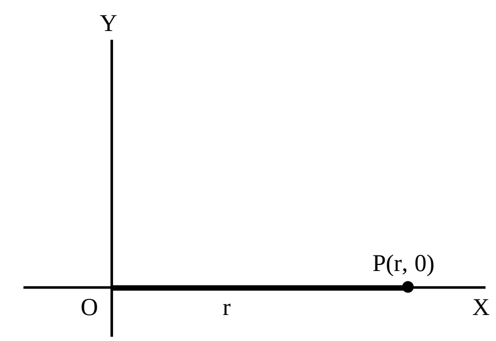
3.9 T-RATIOS OF SOME STANDARD ANGLES :

(i) **Angle of 0 .** Let *OX* be the initial position of the rotating line. Take any point *P* on this line in this position. Then *OP* makes an angle of 0 with *x*-axis. If *OP* = *r* and coordinates of *P* be (*r*, 0), then by definition

$$\cos 0^\circ = \frac{x}{r} = \frac{r}{r} = 1$$

$$\sin 0^\circ = \frac{y}{r} = \frac{0}{r} = 0$$

$$\therefore \quad \tan 0^\circ = \frac{\sin 0^1}{\cos 0^1} = \frac{0}{r} = 0$$



Similarly

$$\sec 0^\circ = \frac{1}{\cos 0^\circ} = \frac{1}{1} = 1$$

$$\text{cosec } 0^\circ = \frac{1}{\sin 0^\circ} = \frac{1}{0} = T$$

$$\cot 0^\circ = \frac{1}{\tan 0^\circ} = \frac{1}{0} = T.$$

(ii) Angle of 30° or π/6 . Rotate the straight line through a positive angle *XOP* of 30°, starting from the initial position *OX*, Make ∠*QOX* = 30° in magnitude. Let *P*(*x*,*y*) be any point in this final position of the rotating line. Draw *PM*⊥*OX* and extend it to meet *OQ* in *Q*. Then evidently Δ *s MOP* and *MOQ* are congruent. Therefore ∠*P* = ∠*Q* = 60°.

Hence Δ *POQ* is equilateral.

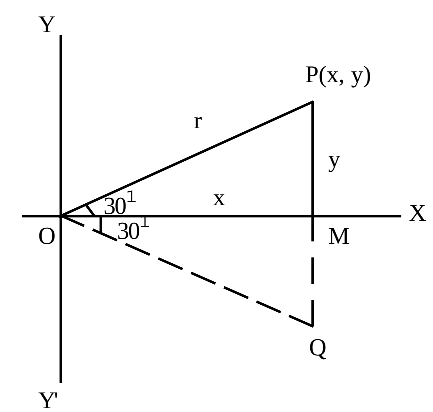
$\therefore \quad MP = \frac{1}{2} \quad PQ = \frac{1}{2} \quad OP = \frac{r}{2}$

From $\Delta\, OMP$,

$$OM^2 = OP^2 - MP^2$$

$$r^2 = 1 - \frac{r^2}{4} = \frac{3r^2}{4}$$

‹ $\quad OM = \frac{\sqrt{3}r}{2}$ [Œ 30° lies in the first quadrant]



Hence $\quad \sin 30^1 = \frac{MP}{OP} = \frac{r}{2r} = \frac{1}{2}$

$$\cos 30^1 = \frac{OM}{OP} = \frac{\sqrt{3}r}{2r} = \frac{\sqrt{3}}{2}$$

$$\tan 30^1 = \frac{MP}{OM} = \frac{r}{2} \cdot \frac{2}{\sqrt{3}r} = \frac{1}{\sqrt{3}}$$

$$\cot 30^1 = \frac{OM}{MP} = \sqrt{3}$$

$$\sec 30^1 = \frac{OP}{OM} = \frac{2}{\sqrt{3}}$$

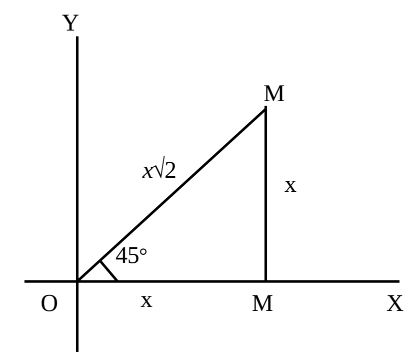
and $\quad \operatorname{cosec} 30^1 = \frac{OP}{MP} = 2$

(iii) **Angle of 45° or π/4.** As before rotate the straight line through a positive angle of 45° with *OX*.

Take $P(x,y)$ any point in the final position. Draw $PM \perp OX$.

Here $OM = MP = x$

$\therefore \quad OP = \sqrt{OM^2 + MP^2} = x\sqrt{2}$



From ΔOMP,

$$\sin 45° = \frac{MP}{OP} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}.$$

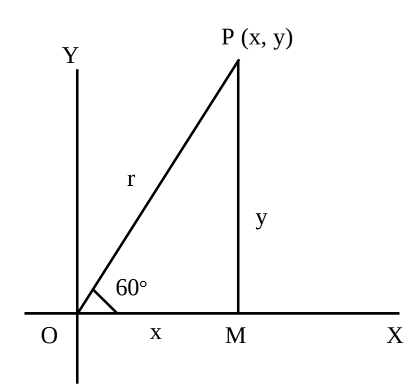
$$\cos 45° = \frac{OM}{OP} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{x}{x} = 1.$$

Similarly, sec 45° = $\sqrt{2}$ = cosec 45° and cot 45° =1.

(iv) **Angle of 60° or π/3.** Rotate the straight line through a positive angle of 60° starting from the initial position OX. Let $P(x,y)$ be any point on the final position of the straight line. Draw $PM \perp OX$. By geometry, $OP = 2OM$

$\therefore \quad r = 2x$ and $MP = y = x\sqrt{3}$



From Δ *OMP*,

$$\sin 60^\circ = \frac{y}{r} = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2}.$$

$$\cos 60^\circ = \frac{x}{r} = \frac{x}{2x} = \frac{1}{2}$$

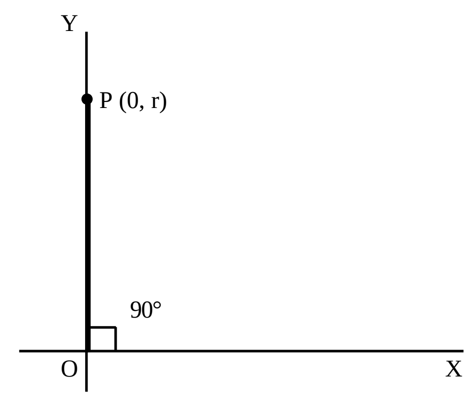
$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}.$$

Similarly $\sec 60^\circ = 2, \ \ \text{cosec}\, 60^\circ = \frac{2}{\sqrt{3}}$

and $\cot 60^\circ = \frac{1}{\sqrt{3}}$

(iv) **Angle of 90° or π/2.** When the rotating straight line makes an angle of 90° with *OX*, it lies along *OY*.

Let *OP* be this position. Here in this position, the coordinates of *P* are (0, *r*)



$\therefore$ y = r , x = 0 $\quad \frac{y}{r} \quad \frac{r}{r}$

$$\sin 90^\circ = \frac{x}{r} = \frac{0}{r} = 1\ .$$

$$\cos 90^\circ = \frac{\sin 90^\circ}{\cos 90^\circ} = \frac{1}{0} = 0$$

$$\tan 90^\circ = \frac{1}{\sin 90^\circ} = \frac{1}{1} = \infty\ .$$

Similarly $\text{cosec}\, 90^\circ = \frac{1}{\cos 90^\circ} = \frac{1}{0} = 1$

$$\sec 90^\circ = \frac{1}{\tan 90^\circ} = \frac{1}{T} = \infty$$

and $\quad \cot 90^\circ = \quad = \quad = 0.$

Table of T-ratios of Standard Angles

Angles(θ)	0°	30°	45°	60°	90°
sinθ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cosθ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tanθ	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	T
cotθ	T	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
secθ	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	T
cosecθ	T	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Example 18. Simplfy

$$\cot^2\frac{\pi}{2}\ \sec^2\frac{\pi}{3}\ \cos\frac{\pi}{2} - 15\sin^2\frac{\pi}{2}\cos\frac{\pi}{4} - 4\cos\frac{\pi}{6}\cos\frac{\pi}{4}\cos\frac{\pi}{2}.$$

Solution. We note that $\cos\frac{\pi}{2} = \cos 90^\circ = 0$, therefore first and third term is 0. Hence there is no need of substituting values in these terms.

$$\text{Given expression} = 0 - 15\sin^2 90^\circ \cos 45^\circ - 0$$

$$= -15 \times (1)^2 \times \frac{1}{\sqrt{2}} = -\frac{15}{\sqrt{2}}.$$

Example 19 (i) Find the value of

$$\mathbf{3\sin^2 30^\circ + 2\tan^2 60^\circ - 5\cos^2 45^\circ}$$

***(ii)* Show that** $\dfrac{2\tan 30^\circ}{1-\tan^2 30^\circ} = \sqrt{3}.$

Solution. Given expression $= 3\sin^2 30^\circ + 2\tan^2 60^\circ - 5\cos^2 45^\circ$

$$= 3\left(\frac{1}{2}\right)^2 + 2(\sqrt{3})^2 - 5\left(\frac{1}{\sqrt{2}}\right)^2$$

$$= \frac{3}{4} + 6 - \frac{5}{2} = \frac{17}{4}$$

(ii) $$\frac{2\tan 30^\circ}{1-\tan^2 30^\circ} = \frac{2\left(\frac{1}{\sqrt{3}}\right)}{1-\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{\frac{2}{\sqrt{3}}}{1-1/3} = \frac{2}{\sqrt{3}} \times \frac{3}{2} = \sqrt{3}.$$

Example 20. Find x if $\tan^2 45 - \cos^2 60 = x \sin 45 \cos 45 \tan 60$

Solution : Putting the values of t-ratios in the given equation, we get

$$(1)^2 - \left(\frac{1}{2}\right)^2 = x \cdot \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \sqrt{3}$$

or $1 - \frac{1}{4} = x\,\frac{\sqrt{3}}{2} \Rightarrow \frac{3}{4} = \frac{\sqrt{3}}{2}x$

$$\Rightarrow x = \frac{\sqrt{3}}{2}$$

Example 21.

If θ is the positive acute angle,

solve the equation, $4\cos^2\theta - 4\sin\theta = 1$

Solution : The given equation is

$4\cos^2\theta - 4\sin\theta = 1$

or $4(1-\sin^2\theta) - 4\sin\theta = 1$

or $4\sin^2\theta + 4\sin\theta - 3 = 0$, which is quadratic in $\sin\theta$,

solving for $\sin\theta$, we get

$\sin\theta = \frac{-4+8}{8} = \frac{1}{2}$ or $\frac{-3}{2}$

As $\sin\theta \le 1$ (numerically), Rejectg the value of $\sin\theta = -3/2$

Therefore, $\sin\theta = \frac{1}{2} = \sin 3^\circ$

$$\Rightarrow \boldsymbol{\theta = 30^0}$$

3.10 SELF ASSESSEMENT QUESTIONS :

1. The angles of a triangle are in AP. The number of degrees in the least is to the number of radions in the greatest is as 60 : II. Find the angle in degrees.

2. A horse is tied to a post by a rope. If the horse moves along a circular path keeping the rope always tight and describes 44 meter when it has traced out 72^0 at the centre, find the length of the rope.

3. Prove that $\sqrt{\frac{1+\cos A}{1-\cos A}} = \operatorname{cosec} A+\cot A$

4. Prove that $\sin^4\theta+\cos^4\theta=1-\sin^2\theta\cos^2\theta$

5. Prove that

$$\frac{\sin\theta}{1+\cos\theta}+\frac{1+\cos\theta}{\sin\theta}=2\operatorname{cosec}\theta$$

6. Show that

$$\frac{1}{\operatorname{cosec}\theta-\cot\theta}-\frac{1}{\sin\theta}=\frac{1}{\sin\theta}\quad\frac{1}{\operatorname{cosec}\theta+\cot\theta}$$

7. Eliminate θ from the equations

$$x\cos\theta+y\sin\theta=a,$$
$$x\sin\theta-y\cos\theta=b$$

8. If cos θ cosec θ =-1 and θ lies in the fourth guadrant, find cos θ & cosec θ.

9. If $\sec\theta=\sqrt{2}$ & θ lines between $\frac{3\pi}{2}$ & 2π, find the value of $\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta}$

10. Prove that

$$\operatorname{cosec}^2\frac{\pi}{6}+\operatorname{cosec}^2-\frac{\pi}{6}\tan^2\frac{\pi}{3}=3$$

11. If $A = 30^0$, Verify that $\cos 3A= 4\cos^3-3\cos A$

3.11 KEY WORDS :

Trigonometry, Quadrant, Angle, Trigonometric Ratios.

3.12 SUGGESTED READINGS :

1. Grewal, B.S. - Engineering Mathematics
2. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.
3. Ramana, B.V. - Higher Engineering Mathematics.
4. Aggarwal, R.S. - Modern Approach of Mathematics.

Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No. :**	**04**
Lesson	**:**	**Trigonometry : T-Ratios of Allied Angles and Summation & Products Formulae**		

4.0 OBJECTIVES :

In this lesson, you will be able to understand

* measurement of angles, trigonometric ratios.
* trigonometric functions & their signs.
* trigonometric functions of standard angles.

4.1 INTRODUCTION :

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90°. The angles $-\theta$, $90^\circ \pm \theta$, $180^\circ \pm \theta$, $360^\circ \pm \theta$ etc., are angles allied to the angle θ, which is being assumed to be expressed in degrees. However if θ is measured in radians (π radians=180°), then angles alied to θ are

$$-\theta,\ \pi/2 \pm \theta,\ \pi \pm \theta,\ 2\pi \pm \theta,\ 2n\pi \pm \theta, \text{etc.}$$

Through the *t*-ratios of allied angles we can find the *t*-ratios of angles of any magnitude. All angles can be represented by $n.90^\circ \pm \theta$,where *n* is zero, an even or an odd integer. Thus, if *n* is zero only $\pm\theta$ angle remains; if *n* is even it may be $180^\circ \pm \theta$ or $360^\circ \pm \theta$ and like that; if *n* is odd then it may be $90^\circ \pm \theta$ or $270^\circ \pm \theta$. We note that except the change of signs depending on the quadrant in which the angle falls, the same *t*-ratios are there when *n* is even and they change to co-ratios if *n* is odd.

4.2 T-RATIOS OF ALLIED ANGLES :- Some important relations between the *t*-ratios of various allied angles are given below.

4.2.1 T-ratios of ($-\theta$) in terms of those of θ, for all values of θ

I. $\sin(-\theta) = -\sin\theta$; $\text{cosec}(\theta)=\text{cosec}\theta$
$\cos(-\theta) = \cos\theta$; $\sec(-\theta) = \sec\theta$
$\tan(-\theta) = -\tan\theta$; $\cot(-\theta) = -\cot\theta$

{ Since $-\theta$ lies in the fourth quadrant, only cos and sec are +ve, all other *t*-ratios are -ve.

4.2.2 T-angle ratios of $(n.90^\circ \pm \theta)$, where *n* is even integer and θ is acute. It is numerically equal to the *t*-ratios of θ. The algebric sign is with reference to the quadrant in which $(n.90^\circ \pm \theta)$ lies.

(a) $\sin(180^\circ-\theta) = +\sin\theta$
$\cos(180^\circ-\theta)=-\cos\theta$
$\tan(180^\circ-\theta)=-\tan\theta$
$\cot(180^\circ-\theta)=-\cot\theta$
$\sec(180^\circ-\theta)=-\sec\theta$
$\text{cosec}(180^\circ-\theta)=+\text{cosec}\theta$

{ Since $(180^\circ-\theta)$ lies in the second quadrant only sin and cosec are +ve, all other *t*-ratios are -ve.

(b) $\sin(180^\circ+\theta) = -\sin\theta$
$\cos(180^\circ+\theta)=-\cos\theta$
$\tan(180^\circ+\theta)=+\tan\theta$
$\cot(180^\circ+\theta)=+\cot\theta$
$\sec(180^\circ+\theta)=-\sec\theta$
$\text{cosec}(180^\circ+\theta)=-\text{cosec}\theta$

{ Since $(180^\circ+\theta)$ lies in the third quadrant, only tan and cot are +ve, all other *t*-ratios are -ve.

(c) $\sin(360^\circ-\theta) = -\sin\theta$
$\cos(360^\circ-\theta)=+\cos\theta$
$\tan(360^\circ-\theta)=-\tan\theta$
$\cot(360^\circ-\theta)=-\cot\theta$
$\sec(360^\circ-\theta)=+\sec\theta$
$\text{cosec}(360^\circ-\theta)=-\text{cosec}\theta$

{ Since $(360^\circ-\theta)$ lies in the fourth quadrant, only cos and sec are + ve, all other *t*-ratios are -ve.

4.2.2 T-angle ratios of (n.90°± θ), where n is even integer and θ is acuteangle.

Here, It is numerically equal to the t-ratios of θ. The algebric sign is with reference to the quadrant in which (n.90° ± θ) lies.

(a) $\sin(180^\circ-\theta) = +\sin\theta$

$\cos(180^\circ-\theta) = -\cos\theta$

$\tan(180^\circ-\theta) = -\tan\theta$

$\cot(180^\circ-\theta) = -\cot\theta$

$\sec(180^\circ-\theta) = -\sec\theta$

$\text{cosec}(180^\circ-\theta) = +\text{cosec}\theta$

Since (180°-θ) lies in the second quadrant only sin and cosec are +ve, all other t-ratios are -ve.

(b) $\sin(180^\circ+\theta) = -\sin\theta$

$\cos(180^\circ+\theta) = -\cos\theta$

$\tan(180^\circ+\theta) = +\tan\theta$

$\cot(180^\circ+\theta) = +\cot\theta$

$\sec(180^\circ+\theta) = -\sec\theta$

$\text{cosec}(180^\circ+\theta) = -\text{cosec}\theta$

Since (180° +θ) lies in the third quadrant, only tan and cot are +ve, all other t-ratios are -ve.

(c) $\sin(360^\circ-\theta) = -\sin\theta$

$\cos(360^\circ-\theta) = +\cos\theta$

$\tan(360^\circ-\theta) = -\tan\theta$

$\cot(360^\circ-\theta) = -\cot\theta$

$\sec(360^\circ-\theta) = +\sec\theta$

$\text{cosec}(360^\circ-\theta) = -\text{cosec}\theta$

Since (360°-θ) lies in the fourth quadrant, only cos and sec are +ve, all other t-ratios are -ve.

(d) $\sin(360^\circ+\theta) = +\sin\theta$

$\cos(360^\circ+\theta) = +\cos\theta$

$\tan(360^\circ+\theta) = +\tan\theta$

$\cot(360^\circ+\theta) = +\cot\theta$

$\sec(360^\circ+\theta) = +\sec\theta$

$\text{cosec}(360^\circ+\theta) = +\text{cosec}\theta$

} Since $360^\circ+\theta$ lies in the first quadrant, all *t*-ratios are +ive.

4.2.3 T - ratios of *(n.* 90° ± θ), where *n* is an odd integer and θ is acute angle.

Here, it is numerically equal to the corresponding co-ratio of θ and vice versa. The algebric sign, as in the previous case is the one applicable to the quadrant in which *(n*. 90°± θ)lies.

(a) $\sin(90^\circ-\theta) = +\cos\theta$

$\cos(90^\circ-\theta) = +\sin\theta$

$\tan(90^\circ-\theta) = +\cot\theta$

$\cot(90^\circ-\theta) = +\tan\theta$

$\sec(90^\circ-\theta) = +\text{cosec}\theta$

$\text{cosec}(90^\circ-\theta) = +\sec\theta$

} Since $(90^\circ-\theta)$ lies in the first quadrant, all the *t*-ratios are +ve but since $90^\circ = 1\times90^\circ$ = odd multiple of 90°, therefore, sin changes to cos, tan changes to cot, sec changes to cosec and *vice versa*.

(b) $\sin(90^\circ+\theta) = \cos\theta$

$\cos(90^\circ+\theta) = -\sin\theta$

$\tan(90^\circ+\theta) = -\cot\theta$

$\cot(90^\circ+\theta) = -\tan\theta$

$\sec(90^\circ+\theta) = -\text{cosec}\theta$

$\text{cosec}(90^\circ-\theta) = +\sec\theta$

} Since $(90^\circ+\theta)$ lies in the second quadrant, only sin and cosec are +ve and all the other *t*-ratios are -ve.

(c) $\sin(270^\circ-\theta) = -\cos\theta$

$\cos(270^\circ-\theta) = -\sin\theta$

$\tan(270^\circ-\theta) = +\cot\theta$

$\cot(270^\circ-\theta) = +\tan\theta$

} Since $(270^\circ-\theta)$ lies in the third quadrant, only tan and cot are +ve and all other *t*-ratios are -ve.

$\sec(270^\circ-\theta) = -\text{cosec}\theta$

$\text{cosec}(180^\circ-\theta) = -\sec\theta$

(d) $\sin(270^\circ+\theta) = -\cos\theta$

$\cos(270^\circ+\theta) = +\sin\theta$

$\tan(270^\circ+\theta) = -\cot\theta$

$\cot(270^\circ+\theta) = -\tan\theta$

$\sec(270^\circ+\theta) = +\text{cosec}\theta$

$\text{cosec}(270^\circ+\theta) = -\sec\theta$

{ Since $(270^\circ+\theta)$ lies in the fourth quadrant, only cos and sec are +ve and all the other *t*-ratios are -ve.

The two important rules to bear in mind are :

1. Any *t*-ratios of an angle expressed as 180° or 360° plus or minus an acute angle, *i.e.*, even number $\times\frac{\pi}{2}$ ± acute angle has numerically the same *t*-ratios as that of the acute angle. The proper signs can be ascertained as per the rules stated below depending on where the revolving line terminates.

II sin	I all
III tan	IV cos

2. Any *t*-ratios of an angle expressed as 90° or 270° plus or minus an acute angle, i.e., odd number $\times\frac{\pi}{2}$ ± acute angle equals numerically the co-*t*-ratios of the acute angle with the plus or minus sign depending upon the quadrant in which the revolving line terminates.

Example 1. Prove that

$$\cos 24^\circ + \cos 55^\circ + \cos 125^\circ + \cos 204^\circ + \cos 300^\circ = \frac{1}{2}$$

Solution. We have

$$\cos 125^\circ = \cos(180^\circ - 55^\circ) = -\cos 55^\circ$$

$$\cos 204^\circ = \cos(180^\circ + 24^\circ) = -\cos 24^\circ$$

$$\cos 300^\circ = \cos(360^\circ - 60^\circ) = \cos(-60^\circ) = \cos 60^\circ = \frac{1}{2}$$

$$\text{L.H.S. } \cos 24^\circ + \cos 55^\circ - \cos 55^\circ - \cos 24^\circ + \frac{1}{2} = \frac{1}{2} = \text{R.H.S.}$$

Example 2. Prove that

$$\cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ = -1.$$

Solution. Now $\cos 510^\circ = \cos(360^\circ + 150^\circ)$

$$= \cos 150^\circ = \cos(180^\circ - 30^\circ)$$

$$= -\cos 30^\circ = -\frac{\sqrt{3}}{2}$$

$$\cos 330^\circ = \cos(360^\circ - 30^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

$$\sin 390^\circ = \sin(360^\circ + 30^\circ) = \sin 30^\circ = \frac{1}{2}$$

and $\cos 120^\circ = \cos(180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2}$

$$\text{L.H.S.} = \cos 510^\circ \cos 330^\circ + \sin 390^\circ \cos 120^\circ$$

$$= \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right) = -\frac{3}{4} - \frac{1}{4} = -1 = \text{R.H.S.}$$

Example 3. Simplify $\dfrac{\cos(90^1+\theta)\ \sec(-\theta)\ \tan(180^1-\theta)}{\sec(360^1-\theta)\ \sin(180^1+\theta)\ \cot(90^1-\theta)}$

Solution. We know that

$$\cos(90^\circ+\theta) = -\sin\theta\ ,\ \sec(-\theta) = \sec\theta$$

$$\tan(180^\circ-\theta) = -\tan\theta\ ,\ \sec(360-\theta) = \sec\theta$$

$$\sin(180^\circ+\theta) = -\sin\theta,\ \cot(90-\theta) = \tan\theta$$

$$\text{Given expression} = \frac{(-\sin\theta)\sec\theta(-\tan\theta)}{\sec\theta(-\sin\theta)\tan\theta} = -1$$

Example 4. Simplify $\dfrac{\sin\theta}{\cos(90^\circ-\theta)} + \dfrac{\tan(-\theta)}{\tan(180^\circ-\theta)} + \dfrac{\sec(180^\circ-\theta)}{\operatorname{cosec}(90^\circ-\theta)}$

Solution. We know that

$\tan(-\theta) = -\tan\theta$, $\cos(90^\circ-\theta) = \sin\theta$

$\sec(180^\circ-\theta)=-\sec\theta$, $\tan(180^\circ-\theta)=-\tan\theta$

$\text{cosec}(90^\circ-\theta)=\sec\theta$

$$\text{Given expression} = \frac{\sin\theta}{\sin\theta} + \frac{\tan\theta}{\tan\theta} - \frac{\sec\theta}{\sec\theta} = 1.$$

Example 5. Find x from the equation :

$$\text{cosec}(90^\circ+A)+x \cos A \cot(90^\circ+A) = \sin(90^\circ+A).$$

Solution. We know that

$$\text{cosec}(90^\circ+A) = \sec A =$$

$$\cot(90^\circ+A) = -\tan A = -\frac{1}{\cos A}$$

$$\sin(90^\circ+A) = \cos A \quad \frac{\sin A}{\cos A}$$

Substituting these values in the given equation, we get

$$\frac{1}{\cos A} + x \cos A \left(-\frac{\sin A}{\cos A}\right) = \cos A$$

$$x \sin A = \frac{1}{\cos A} - \cos A = \frac{1-\cos^2 A}{\cos A} = \frac{\sin^2 A}{\cos A}$$

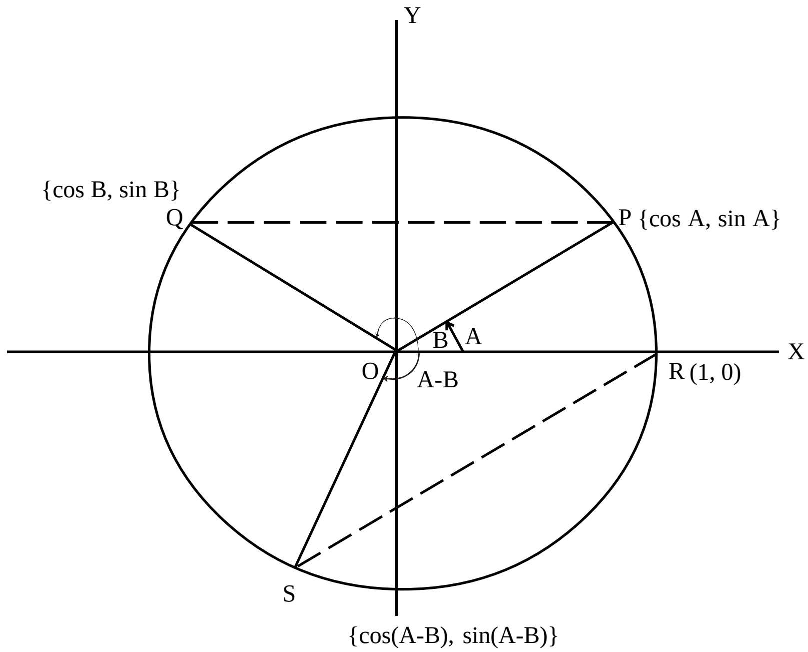
$$x = \frac{\sin^2 A}{\cos A} \times \frac{1}{\sin A} = \tan A.$$

4.3 COSINE OF DIFFERENCE OF TWO ANGLES :

To show that $\cos(A-B) = \cos A \cos B + \sin A \sin B$.

Let us suppose that a revolving line starting from *OX* trace out angle *A* and *B* so that terminal side of the angle *A* and *B* cuts the circle in *P* and *Q* respectively. Obviously co-ordinates of *P* and *Q* are $(\cos A, \sin A)$ and $(\cos B, \sin B)$ respectively. Again starting from *OX* and revolving clockwise trace out an angle *ROS* equal to angle *POQ* in magnitude.

$$\therefore \quad \angle ROS = -(POQ) = -(B-A) = A-B$$



Since in the circle of unit radius centered at origin, co-ordinates of a point where terminal side of the angle cuts the circle are given by cosine and sine of the angle.

$\therefore$ S is $[\cos(A-B), \sin(A-B)]$

Since equal chords subetend equal angles at the centre of the circle therefore,

Length of chord PQ = length of chord RS

$$\sqrt{(\cos A-\cos B)^2+(\sin A-\sin B)^2}$$

$$=\{1-\cos(A-B)\}^2+\{(0-\sin(A-B)\}^2$$

(By distance formula)

By squaring, we have

$$\cos^2 A+\cos^2 B-2\cos A\cos B+\sin^2 A+\sin^2 B-2\sin A\sin B$$

$$=1+\cos^2(A-B)-2\cos(A-B)+\sin^2(A-B)$$

$\Rightarrow$ $\cos(A-B) = \cos A \cos B+\sin A \sin B$

4.4. ADDITION FORMULAE :

To show that :

(i) $\sin(A+B)=\sin A \cos B + \cos A \sin B$

(ii) $\cos(A+B)=\cos A \cos B - \sin A \sin B$

(iii) $\tan(A+B) = \dfrac{\tan A + \tan B}{1-\tan A \tan B}$.

Proof. *(i)* We know

$$\cos(\theta - \phi)=\cos\theta \cos\phi + \sin\theta \sin\phi$$

Put $\theta = 90^\circ - A$ and $\phi = B$ in (i),

$$\cos(90^\circ - A - B) = \cos(90^\circ - A)\cos B + \sin(90^\circ - A)\sin B$$

$$\cos(90^\circ - (A+B)) = \sin A \cos B + \cos A \cos B$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

(ii) Taking $\theta = A$ and $\phi = -B$ in (i), we get

$$\cos(A+B) = \cos A \cos(-B) + \sin A \sin(-B)$$

$$= \cos A \cos B - \sin A \sin B$$

(iii) Divide the results obtained in part *(i)* and *(ii)*,

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}.$$

Dividing numerator and denominator on R.H.S. by cos A cos B, we get

$$\frac{\sin(A+B)}{\cos(A+B)} = \frac{\tan A + \tan B}{1-\tan A \tan B}.$$

4.5. SUBTRACTION FORMULAE :

To show that :

(i) $\cos(A-B)= \cos A \cos B + \sin A \sin B$

(ii) $\sin(A-B) = \sin A \cos B - \cos A \sin B$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1+\tan A \tan B}$.

Proof : *(i)* Already proved in 4.3

(ii) Replace A by $\frac{\pi}{2}+A$ in *(i)* part

$$\cos\left(\frac{\pi}{2}+A-B\right) = \cos\left(\frac{\pi}{2}+A\right)\cos B+\sin\left(\frac{\pi}{2}+A\right)\sin B$$

$$-\sin(A-B) = -\sin A \cos B + \cos A \sin B$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B$$

(iii) Dividing the results of part *(i)* by that of *(ii)*

$$\frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

Dividing numerator and denominator on R.H.S. by cos A cosB, we have

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A \tan B}.$$

4.5.1 Prove the following :

(i) $\tan(45^\circ+A) = \dfrac{1+\tan A}{1-\tan A}$

(ii) $\tan(45^\circ-A) = \dfrac{1-\tan A}{1+\tan A}$

(iii) $\cot(A+B) = \dfrac{\cot A \cot B-1}{\cot B + \cot A}$

(iv) $\cot(A-B) = \dfrac{\cot A \cot B+1}{\cot B - \cot A}$

Proof.

(i) $\tan(45^\circ+A) = \dfrac{\tan 45^\circ + \tan A}{1-\tan 45^\circ \tan A} = \dfrac{1+\tan A}{1-\tan A}$.

(ii) $\tan(45^\circ-A) = \dfrac{\tan 45^\circ - \tan A}{1+\tan 45^\circ \tan A} = \dfrac{1-\tan A}{1+\tan A}$.

(iii) $\cot(A+B) = \dfrac{1}{\tan(A+B)} = \dfrac{1-\tan A \tan B}{\tan A + \tan B}$

$$= \frac{1-\dfrac{1}{\cot A}\dfrac{1}{\cot B}}{\dfrac{1}{\cot A} + \dfrac{1}{\cot B}}$$

$$= \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

(iv) $\cot(A-B) = \dfrac{1}{\tan(A-B)} = \dfrac{1+\tan A \tan B}{\tan A - \tan B}$

$$= \frac{1+\dfrac{1}{\cot A}\dfrac{1}{\cot B}}{\dfrac{1}{\cot A} - \dfrac{1}{\cot B}}$$

$$= \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

4.6 TWO IMPORTANT THEOREMS :

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

Proof.

(i) $\sin(A+B) \cdot \sin(A-B)$

$= (\sin A \cos B + \cos A \sin B)(\sin A \cos B - \cos A \sin B)$

$= \sin^2 A \cos^2 B - \cos^2 A \sin^2 B$

$= \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A \sin^2 B - \sin^2 B + \sin^2 B \sin^2 A$

$= \sin^2 A - \sin^2 B$ (first part is proved)

$= 1-\cos^2 A - (1-\cos^2 B)$

$= \cos^2 B - \cos^2 A$ (second part is proved)

(ii) $\cos(A+B)\cos(A-B)$

$=(\cos A\cos B-\sin A\sin B)(\cos A\cos B+\sin A\sin B)$

$=\cos^2 A\cos^2 B-\sin^2 A\sin^2 B$

$=\cos^2 A(1-\sin^2 B)-(1-\cos^2 A)\sin^2 B$

$=\cos^2 A-\cos^2 A\sin^2 B-\sin^2 B+\sin^2 B\cos^2 A$

$=\cos^2 A-\sin^2 B$ (first part is proved)

$=1-\sin^2 A-(1-\cos^2 B)$

$=\cos^2 B-\sin^2 A$ (second part is proved)

4.7 TRANSFORMATION OF A PRODUCT INTO SUM OR DIFFERENCE:

To prove that

(i) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

(ii) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

(iii) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

(iv) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Proof. We know that

$\sin(A+B)=\sin A\cos B+\cos A\sin B$ ...(1)

$\sin(A-B)=\sin A\cos B-\cos A\sin B$ ...(2)

$\cos(A+B)=\cos A\cos B-\sin A\sin B$ ...(3)

$\cos(A-B)=\cos A\cos B+\sin A\sin B$ ...(4)

(1)+(2) gives $\sin(A+B)+\sin(A-B)=2\sin A\cos B$

(1)-(2) gives $\sin(A+B)-\sin(A-B)=2\cos A\sin B$

(3)+(4) gives $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

(4)-(3) gives $\cos(A-B)-\cos(A+B)=2\sin A\sin B$.

4.8 TRANSFORMATION OF A SUM OR DIFFERENCE INTO PRODUCT :

To Prove that

(i) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(ii) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(iii) $\cos C+\cos D= 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(iv) $\cos D-\cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$.

Proof.

Taking $A+B=C$ and $A-B=D$ so that

$$A = \frac{C+D}{2} \quad \text{and } B = \frac{C-D}{2}$$

From formulae deduced in art. V, we have

$$\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$$

$$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$$

$$\cos D - \cos C = 2\sin\frac{C+D}{2}\sin\frac{C-D}{2}.$$

4.9 DOUBLE ANGLE FORMULAE :

To Prove that

(i) $\sin 2A = 2\sin A\cos A$

(ii) $\sin 2A = \frac{2\tan A}{1+\tan^2 A}$

(iii) $\cos 2A= \cos^2 A-\sin^2 A$

(iv) $\cos 2A = 2\cos^2 A-1$

(v) $\cos 2A = 1 - 2\sin^2 A$

(vi) $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(vii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Proof.

(i) $\sin 2A = \sin(A+A) = \sin A \cos A + \cos A \sin A$

$= 2 \sin A \cos A$

(ii) $\sin 2A = \dfrac{2 \sin A \cos A}{1} = \dfrac{2 \sin A \cos A}{\cos^2 A + \sin^2 A}$

$= \dfrac{\dfrac{2 \sin A \cos A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \dfrac{2\tan A}{1+\tan^2 A}$

(iii) $\cos 2A = \cos(A+A) = \cos A \cos A - \sin A \sin A$

$= \cos^2 A - \sin^2 A$

(iv) $\cos 2A = \cos^2 A - \sin^2 A = \cos^2 A - (1-\cos^2 A) = 2\cos^2 A - 1$

(v) $\cos 2A = \cos^2 A - \sin^2 A = 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$

(vi) $\cos 2A = \dfrac{\cos^2 A - \sin^2 A}{\cos^2 A + \sin^2 A} = \dfrac{\dfrac{\cos^2 A - \sin^2 A}{\cos^2 A}}{\dfrac{\cos^2 A + \sin^2 A}{\cos^2 A}} = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

(vii) $\tan 2A = \tan(A+A) = \dfrac{\tan A + \tan A}{1 - \tan A \tan A} = \dfrac{2\tan A}{1-\tan^2 A}$.

Note. $\cos 2A = 1 - 2\sin^2 A \Rightarrow 2\sin^2 A = 1 - \cos 2A$

$\cos 2A = 2\cos^2 A - 1 \Rightarrow 2\cos^2 A = 1 + \cos 2A$

These formulae are often used for solving problems in triginometry.

4.10 T-RATIOS of 3A :

To Prove that :

(i) $\sin 3A = 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$.

Proof.

(i) $\sin 3A = \sin(A+2A) = \sin A \cos 2A + \cos A \sin 2A$

$= \sin A\,(1-2\sin^2 A) + \cos A \,.\, 2\sin A \cos A$

$= \sin A - 2\sin^3 A + 2\sin A(1-\sin^2 A)$

$= 3\sin A - 4\sin^3 A$

(ii) $\cos 3A = \cos(A+2A) = \cos A \cos 2A - \sin A \sin 2A$

$= \cos A\,(2\cos^2 A - 1) - \sin A \;\; 2\sin A \cos A$

$= 2\cos^3 A - \cos A - 2\cos A\,(1-\cos^2 A)$

$= 4\cos^3 A - 3\cos A$

(iii) $\tan 3A = \tan(A+2A) = \dfrac{\tan A + \tan 2A}{1 - \tan A \tan 2A}$

$$= \frac{\tan A + \dfrac{2\tan A}{1-\tan^2 A}}{1 - \tan A\, \dfrac{2\tan A}{1-\tan^2 A}}$$

$$= \frac{\tan A - \tan^3 A + 2\tan A}{1 - \tan^2 A - 2\tan^2 A} = \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$

4.11 TWO IMPORTANT RESULTS :

To find the value of *(i)* sin 18° *(ii)* cos 36°

Proof. *(i)* Let $A = 18^\circ$ (i)

$5A = 90^\circ \quad \Rightarrow \quad 2A + 3A = 90^\circ$

or $2A = 90^\circ - 3A \Rightarrow \sin 2A = \sin(90^\circ - 3A)$

which gives $2 \sin A \cos A = \cos 3A = 4 \cos^3 A - 3 \cos A$

or $2 \sin A = 4(1 - \sin^2 A) - 3$

or $4 \sin^2 A + 2 \sin A - 1 = 0$

or $$\sin A = \frac{-2 \pm \sqrt{4+16}}{8} = \frac{-2 \pm 2\sqrt{5}}{8}$$

$$\sin 18^\circ = \frac{-1 \pm \sqrt{5}}{4}$$

But $\sin 18^\circ \neq \frac{-1-\sqrt{5}}{4}$ as 18° lies in the first quadrant

$$\therefore \quad \sin 18 = \frac{\sqrt{5}-1}{4}$$

Cor. 1. Now $\cos^2 18^\circ = 1 - \sin^2 18^\circ$

$$= 1 - \left(\frac{\sqrt{5}-1}{4}\right)^2 = 1 - \frac{5+1-2\sqrt{5}}{16}$$

$$= \frac{10 + 2\sqrt{5}}{16}$$

$$\cos 18 = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$$

Rejecting $\cos 18 = -\frac{\sqrt{10 + 2\sqrt{5}}}{4}$

Cor. 2. $\sin 72^\circ = (\sin 90^\circ - 18^\circ) = \cos 18^\circ = \frac{\sqrt{10 + 2\sqrt{5}}}{4}$

$\cos 72^\circ = \cos(90^\circ - 18^\circ) = \sin 18^\circ = \frac{\sqrt{5}-1}{4}$

(ii) We know $\cos 2A = 1 - 2\sin^2 A$

$\cos 36^\circ = 1 - 2 \sin^2 18^\circ$ (Taking $A = 18^\circ$)

$$\cos 36^\circ = 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 = \frac{16 - 2(5+1-2\sqrt{5})}{16}$$

$$= \frac{4 + 4\sqrt{5}}{16} = \frac{\sqrt{5}+1}{4}$$

Cor.1. $\sin 36^\circ = \sqrt{1-\cos^2 36^\circ} = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2}$

$$= \frac{\sqrt{10-2\sqrt{5}}}{4}$$

Cor.2. $\sin 54^\circ = \sin(90^\circ-36^\circ) = \cos 36^\circ = \frac{\sqrt{5}+1}{4}$

$\cos 54^\circ = \cos(90^\circ-36^\circ) = \sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$

Note. *We can also obtain the value of cos 36. independently by taking A=36 and pro*ceeding *in the same way as in Art. 4.11*

4.12 T-RATIOS OF "A" IN TERMS OF THOSE OF A/2 :

In article 4.9 we have obtained *t*-ratios of 2*A* in terms of *A*. Taking $2A = \theta$ and

$A = \frac{\theta}{2}$ in all these, we get

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$\frac{\theta}{2}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}$

(iii) $\cos\theta = 1-2\sin^2\frac{\theta}{2}$

(iv) $\cos\theta = 2\cos^2\frac{\theta}{2} - 1$

(v) $\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(vi) $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(vii) $\tan\theta = \frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

Example 6. Find the value of

(i) $\sin 15^\circ$ (ii) $\cos 15^\circ$ (iii) $\tan 15^\circ$

Solution. $\sin 15^\circ = \sin(45^\circ-30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

cos 15º = cos (45º-30º)= cos 45º cos 30º +sin 45º sin 30º

$$= \frac{1}{\sqrt{2}} \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\tan 15^\circ = \tan(45^\circ-30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1+\tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3}-1)(\sqrt{3}+1)} = \frac{3+1-2\sqrt{3}}{3-1} = 2-\sqrt{3}.$$

Example 7. If A and B are positive acute angles such that $\sin A = \frac{1}{\sqrt{10}}$, $\sin B = \frac{1}{\sqrt{5}}$ prove that $A+B = \frac{\pi}{4}$.

Solution. $\cos A = \sqrt{1-\sin^2 A}$

$$= \sqrt{1-\frac{1}{10}} = \frac{3}{\sqrt{10}}$$

$\cos B = 1-\sin^2 B$

$$= \sqrt{1-\frac{1}{5}} = \frac{2}{\sqrt{5}}$$

[cos *A* and cos *B* and + ve as *A* and *B* are acute angles]

By formula

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A+B) = \frac{3}{\sqrt{10}} \frac{2}{\sqrt{5}} \quad \frac{1}{\sqrt{10}} \frac{1}{\sqrt{5}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}}$$

$$\therefore A+B = \frac{\pi}{4}$$

Example 8. Prove that tan 70º = tan 20º + 2 tan 50º.

Solution. tan 70º = tan(20º+50º)

or $\quad \tan 70^\circ = \frac{\tan 20^\circ + \tan 50^\circ}{1-\tan 20^\circ \tan 50^\circ}$

or $\quad \tan 70^\circ - \tan 70^\circ \tan 20^\circ \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$

or $\tan 70^\circ - \tan(90^\circ - 20^\circ) \tan 20^\circ \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$

or $\tan 70^\circ - \cot 20^\circ \tan 20^\circ \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$

or $\tan 70^\circ - \tan 50^\circ = \tan 20^\circ + \tan 50^\circ$ [Œ $\cot 20^\circ \tan 20^\circ = 1$]

Hence, $\tan 70^\circ = \tan 20^\circ + 2 \tan 50^\circ$

Example 9. Express as a sum or difference :

(i) $2 \sin 3\theta \cos\theta$

(ii) $2 \cos 2\theta \sin 4\theta$

(iii) $\cos 3\theta \cos 5\theta$

(iv) $\sin \frac{\theta}{2} \sin \frac{3\theta}{2}$.

Solution.

(i) $2 \sin 3\theta \cos\theta = \sin(3\theta+\theta) + \sin(3\theta-\theta) = \sin 4\theta + \sin 2\theta$

(ii) $2 \cos 2\theta \sin 4\theta = 2 \sin 4\theta \cos 2\theta = \sin(4\theta+2\theta) + \sin(4\theta-2\theta)$

$= \sin 6\theta + \sin 2\theta$ | using $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

(iii) $\cos 3\theta \cos 5\theta = \frac{1}{2} (2 \cos 5\theta \cos 3\theta)$ | using $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

$= \frac{1}{2} [\cos(5\theta+3\theta) + \cos(5\theta-3\theta)]$

$= \frac{1}{2} [\cos 8\theta + \cos 2\theta]$

(iv) $\sin\frac{\theta}{2} \sin\frac{3\theta}{2} = \frac{1}{2} \left[2 \sin\frac{3\theta}{2} \sin\frac{\theta}{2}\right]$

$= \frac{1}{2}\left[\cos\left(\frac{3\theta}{2} - \frac{\theta}{2}\right) - \cos\left(\frac{3\theta}{2} + \frac{\theta}{2}\right)\right]$

$= \frac{1}{2} [\cos\theta - \cos 2\theta]$

Example 10. Show that $\sin 20 \sin 40 \sin 80 = \frac{\sqrt{3}}{8}$.

Solution. L.H.S. $= \sin 20 \sin 40 \sin 80 = \frac{1}{2}[2\sin 80 \sin 40] \sin 20$

$= \frac{1}{2} [\cos(80-40) - \cos(80+40)] \sin 20$

$$= \frac{1}{2}\,[\cos 40 \sin 20 + \frac{1}{2}\sin 20] = \frac{1}{2}\cos 40 \sin 20 + \frac{1}{4}\sin 20$$

$$= \frac{1}{4}\,2\cos 40 \sin 20 + \frac{1}{4}\sin 20$$

$$= \frac{1}{4}[\sin(40+20) - \sin(40-20)] + \frac{1}{4}\sin 20 = \frac{1}{4}\sin 60$$

$$= \frac{1}{4}\,\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}\,.$$

Example 11. Prove that

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}$$

OR

$$\sin 36^o \sin 72^o \sin 108^o \sin 144^o = \frac{5}{16}$$

Solution.

L.H.S. $= \sin 36^o \sin(90^o-18^o) \sin(90^o+18^o) \sin(180^o-36^o)$

$$= \sin 36^o \cos 18^o \cos 18^o \sin 36^o = \sin^2 36^o \cos^2 18^o$$

$$= (1-\cos^2 36^o)\,(1-\sin^2 18^o)$$

$$= \left[1-\left(\frac{\sqrt{5}+1}{4}\right)^2\right]\left[1-\left(\frac{\sqrt{5}-1}{4}\right)^2\right]$$

$$= \left(1-\frac{5+1+2\sqrt{5}}{16}\right)\left(1-\frac{5+1-2\sqrt{5}}{16}\right)$$

$$= \left(\frac{16-6-2\sqrt{5}}{16}\right)\left(\frac{16-6+2\sqrt{5}}{16}\right)$$

$$= \frac{(10-2\sqrt{5})(10+2\sqrt{5})}{256} = \frac{100-20}{256} = \frac{5}{16}\,.$$

Example 12. Prove that

$$\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{16}$$

Solution. Converting the angles into degrees and writing

L.H.S. = $\cos 12^o \cos 24^o \cos 48^o \cos 84^o$

$$= \frac{1}{2}(2\cos 48^o \cos 12^o)\frac{1}{2}(2\cos 84^o \cos 24^o)$$

$$= \frac{1}{2}[\cos(48+12)+\cos(48-12)]\ \frac{1}{2}\ [\cos(84+24)+\cos(84-24)]$$

L.H.S. $= \frac{1}{4}[\cos 60+\cos 36][\cos 108+\cos 60]$

$$= \frac{1}{4}\left[\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right]\left[\cos(90+18)+\frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{5}+3}{4}\right]\left[-\sin 18+\frac{1}{2}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{5}+3}{4}\right]\left[\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{5}+3}{4}\right]\left[\frac{3-\sqrt{5}}{4}\right] = \frac{9-5}{64} = \frac{1}{16}.$$

4.13 SELF ASSESSMENT QUESTIONS :

1. Evaluate : (1) $\text{Cos } 1050^0$ (ii) $\text{Sin } 300^0$,

(iii) $\tan(-1575^0)$

2. Prove that

$$\text{Cot}(-405^0)\tan 315^0+\tan(-585^0)\text{Cot}(495^0)=2$$

3. Simplify :

$$\frac{\text{Sin}(180^0+\theta)\text{Sec}(270^0+\theta^0)\tan(90^0+\theta)}{\text{Cos}(180-\theta)\text{Cot}(360-\theta^0)\text{Cos}(270+\theta)}$$

4. Solve for x the following equations :

$$\text{Sec}(90^0+A)+x.\sin A\ \tan(90^0+A)=\text{Cos}(90+A).$$

5. Show that
$$\frac{\tan\left(\frac{\pi}{4}+A\right)+\tan\left(\frac{\pi}{4}-A\right)}{\tan\left(\frac{\pi}{4}+A\right)-\tan\left(\frac{\pi}{4}-A\right)} = \text{Cosec } 2A$$

6. Evaluate : (i) $\tan 75^0+\text{Cot } 75^0$, (iii) $\text{Sin } 75^0+\text{Cos } 75^0$

7. Prove that

$$\text{Cosec } (45^0+\theta) \text{ Cosec } (45^0-\theta) = 2 \sec 2\theta$$

8. Prove that

$$\text{Cosec } 20^0 \text{ Cos } 40^0 \text{ Cos } 60^0 \text{ Cos } 80^0 = 1$$

9. Prove that

$$\text{Sin } 20^0 \text{ Sin } 40^0 \text{ Sin } 60^0 \text{ Sin } 80^0 = 3/16$$

10. Find the value of sin 2x and Cos 2x, when

(i) $\text{Sin } x = \frac{4}{5}$ (ii) $\text{Cos } x = 5/13$

4.14 KEY WORDS :

T-Ratios of Allied Angles, Summation & Product Formula.

4.15 SUGGESTED READINGS :

1. Grewal, B.S. - Engineering Mathematics
2. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.
3. Ramana, B.V. - Higher Engineering Mathematics.
4. Aggarwal, R.S. - Modern Approach of Mathematics.

E E E

Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No.**	**: 05**
Lesson	**:**	**Analytical Plane Geometry**		

5.0 OBJECTIVES :

In this lesson, you will be able to understand

* Co-ordinates, distance between two points, area of a tringle, collinearity of three points, locus of a point.
* Straight line and different forms of straight lines & their applications in solving problems.

5.1 INTRODUCTION :

Co-ordinate Geometry or Analytical plane Geometry is that branch of geometry which defines the position of a point in a plane by an ordered pair of algebraic numbers called co-ordinates. It is also called the wedding of Algebra and Geometry.

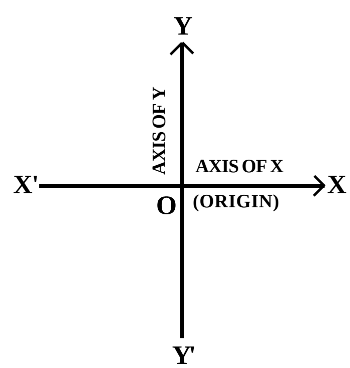
5.2 DEFINITIONS :

5.2.1 CO-ORDINATE AXES

Let $X'OX$ and $Y'OY$ be two mutually perpendicular lines taken as axes whose positive directions are shown by arrows on the axes. These lines are called the **Co-ordinate Axes.**

$X'OX$ is called the **x-axis.**

$Y'OY$ is called the **y-axis.**



As the axes taken are mutually at right angles, they are called the Rectangular Axis. The point O (i.e., the point of intersection of the axes is called the *Origin.*

5.2.2 QUARDRANTS

The axes divide the plane area into four parts called quadrants.

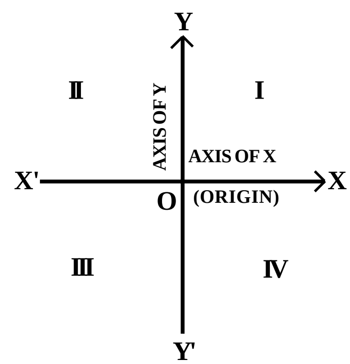
XOY is the first quadrant.

YOX' is the second quadrant.

X'OY' is the third quadrant.

Y'OX is the fourth quadrant.

These quadrants cannot be numbered as *I, II, III* and *IV* in any other order.

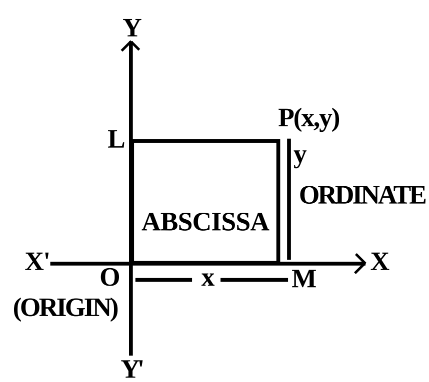


5.2.3 CO-ORDINATES OF A POINT

Let *P* be any point in the plane. From *P* draw *PL*, *PM* b on the *y*-axis and *x*-axis respectively. Then length *LP* is called the *x*-axis co-ordinate or the abscissa of point *P* and *MP* is called the *y* co-ordinate or the ordinate of point *P*. Point whose abscissa in *x* and ordinate is *y* is called the point (*x, y*).

Note :

(i) While writing co-ordinates of a point, the x-co-ordinates is written first and then the y-co-ordinate, separated by comma.



(ii) The co-ordinates of points lying in the first, second, third and fourth quadrants have the signs as (+,+), (-,+), (-, -), (+, -) respectively.

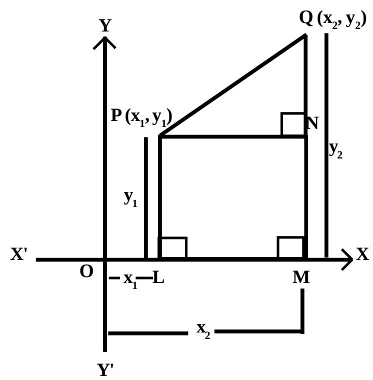
(iii) x-co-ordinate or abscissa of all the points on the y-axis is zero. Thus, any point on the y-axis is of the form (0, y).

(iv) y-co-ordinate or ordinate of all the points on the x-axis is zero. Thus, any point on the x-axis is of the form (x,0).

(v) *co-ordinates of origin are (0,0) as it lies on both the axes.*

5.3 DISTANCE BETWEEN TWO POINTS :

To find the distance between two given points. P (x_1,y_1) and Q (x_2, y_2)



Let $P(x_1,y_1)$, $Q(x_2,y_2)$ be the given points. From points P and Q draw PL, QM, ⊥s on $X'OX$.

Then $OL = x_1$, $LP = y_1$ $OM = x_2$, $MQ = y_2$

Draw PN perpendicular to MQ. From the right- angled ΔPNQ, we have

$$(PQ)^2 = (PN)^2 + (NQ)^2 = (LM)^2 + (NQ)^2 = (OM-OL)^2 + (MQ-MN)^2$$

$$= (OM-OL)^2 + (MQ-LP)^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

or PQ = $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ or $\sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$.

Cor.1. Distance of the point (x_1, y_1) from origin is given by

$$\sqrt{(x_1-0)^2 + (y_1-0)^2} = \sqrt{x_1^2 + y_1^2}.$$

Cor. 2. If PL and QM are perpendiculars from P and Q upon $X'OX$, then, LM is called Projection of PQ upon x-axis. Thus projection of PQ upon x-axis = $LM=OM-OL = x_2-x_1$.

Similarly projection of PQ upon y-axis = y_2-y_1.

Note. In questions relating to geometrical figures, take the given vertices in the given order and prove the desired as under :

(i) **For an isosceles triangle** -Prove that at least two sides are equal.

(ii) **For an equilateral triangle** -Prove that three sides are equal.

(iii) **For a right-angled triangle** -Prove that the sum of the squares of two sides is equal to the square of the third side.

(iv) **For collinear points** -Prove that the sum of the distances between two point-pairs is equal to the distance between the third point pair.

(v) **For a square** -Prove that the four sides are equal, two diagonals are equal.

(vi) **For a rectangle** -Prove that the opposite sides are equal and two diagonals are equal.

Example 1. Show that the points (a, b+c), (b, c+a), (c, a+b) are collinear.

Solution : Let us denote the points as

$$A(a,\ b+c),\ B(b,\ c+a),\ C(c,\ a+b).$$

By using distance formula, we have

$$AB = \sqrt{(b-a)^2+(c+a-b-c)^2}$$

$$= \sqrt{(b-a)^2+(a-b)^2} = \sqrt{2}(a-b)$$

$$BC = \sqrt{(c-b)^2 + (a+b-c-a)^2}$$

$$= \sqrt{(c-b)^2 + (b-c)^2} = \sqrt{2}(b-c)$$

$$AC = \sqrt{(c-a)^2 + (a+b-b-c)^2}$$

$$= \sqrt{(c-a)^2 + (a-c)^2} = \sqrt{2}(a-c)$$

$$AB+BC = \sqrt{2}\,(a-b+b-c) = \sqrt{2}(a-c) = AC.$$

Therefore, points are collinear.

Example 2. P, Q and S are the points $(at^2, 2at)$, $(a/t^2, -2a/t)$ and (a, 0) respectively, show that

$$\frac{1}{SQ} + \frac{1}{SP} = \frac{1}{a}.$$

Solution : Using distance formula,

$$SP = \sqrt{(at^2-a)^2 + (2at-0)^2} = \sqrt{a^2(t^2-1)^2 + 4a^2t^2}$$

$$= a\sqrt{(t^2-1)^2 + 4t^2} = a\sqrt{(t^2+1)^2}$$

$$SP = a(t^2+1) \qquad \text{.................. (1)}$$

$$SQ = \sqrt{\left(a - \frac{a}{t^2}\right)^2 + \left(0 + \frac{2a}{t}\right)^2}$$

$$= a\sqrt{\frac{(t^2-1)^2+4t^2}{t^4}} = \frac{a(t^2+1)}{t^2} \qquad \text{.................. (2)}$$

Adding (1) and (2), $\frac{1}{SQ} + \frac{1}{SP} = \left(\frac{1}{a(t^2+1)} + \frac{t^2}{a(t^2+1)}\right)$

$$= \frac{1+t^2}{a(t^2+1)} = \frac{1}{a} \text{ (Proved)}$$

Example 3. The two vertices of an equilateral triangle are (0,0) and (3, 3), find the third vertex.

Solution. Let A(0,0), B(3, 3) and C(x,y) be vertices of an equilateral triangle ABC, therefore AB=AC=BC gives

$$\sqrt{(3-0)^2+(\sqrt{3}-0)^2} = \sqrt{(x-0)^2+(y-0)^2}$$

$$= \sqrt{(x-3)^2+(y-\sqrt{3})^2}$$

or $9+3 = x^2+y^2 = x^2+y^2-6x-2 \quad 3y+12$ ...(1)

From Ist two members of (1), $x^2+y^2=12$ ...(2)

From last two members, $6x+2 \quad 3y=12$ ...(3)

Eliminating y between (2) and (3), we have $4x^2-12x=0$

i.e. $4x(x-3)=0$ which gives $x=0$ or 3

If $x=0$, $y= 2 \quad 3$ (from 3) and if $x=3$, $y= - \quad 3$

Co-ordinates of C are $(0,2 \quad 3)$ or $(3, - \quad 3)$.

Example 4. Find the abscissa of points whose ordinate is 4 and which are at a distance of 5 units from (5,0)

Solution. Let $A(x,4)$, $B(5,0)$

Then $(AB)^2 = (5)^2$

$(x-5)^2 + (4-0)^2 = 25$

$x^2+25-10x+16 = 25$

$x^2-10x+16 = 0$

$x^2-8x-2x+16 = 0$

$x(x-8)-2(x-8) = 0$

$(x-2)(x-8) = 0$

$x=2$ or $x=8$

Example 5. The point (x,y) is equidistant from the points (a+b, b-a) and (a-b, a+b), prove that bx=ay.

Solution. Let $A(x,y)$ and $B(a+b, b-a)$ and $C(a-b, a+b)$

Then $(BA)^2 = (CA)^2$

gives $(a+b-x)^2 + (b-a-y)^2 = (a-b-x)^2 + (a+b-y)^2$ or

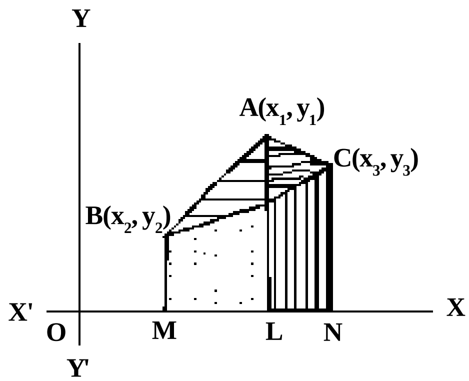
$a^2+b^2+x^2+2ab-2bx-2ax+b^2+a^2+y^2-2ba-2by+2ay=a^2+b^2+x^2-2ab-2ax+2bx+a^2+b^2+y^2+2ab-2ay-2by$

or $-2bx-2ax-2by+2ay = -2ax+2bx-2ay-2by$

or $4ay = 4bx$ or $ay = bx$

5.4 AREA OF A TRIANGLE :

To find the area of a triangle where co-ordinates of vertices are given.



Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be the vertices of a triangle *ABC*. From *A*, *B*, *C* draw *AL*, *BM*, *CN* perpendiculars on the axis of *x*. Area of *triangle ABC* is

= area of trapezium *ABML* + trap. *ALNC* - trap. *BMNC*

$= \frac{1}{2}(MB+LA)\,ML + \frac{1}{2}(LA+NC)LN - \frac{1}{2}(MB+NC)MN$

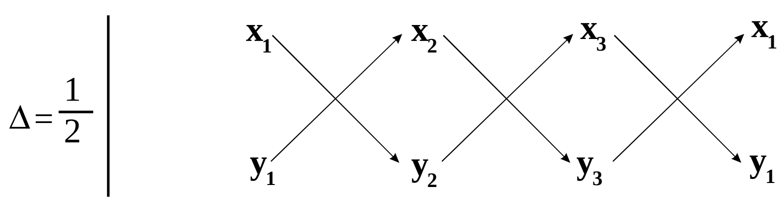
$= \frac{1}{2}(MB+LA)(OL-OM)+\frac{1}{2}(LA+NC)(ON-OL)-\frac{1}{2}(MB+NC)(ON-\mathrm{OM})$

$= \frac{1}{2}(y_2+y_1)(x_1-x_2)+\frac{1}{2}(y_1+y_3)(x_3-x_1)-\frac{1}{2}(y_2+y_3)(x_3-x_2)$

$\Delta = \frac{1}{2}[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)]$

(After multiplying the products and cancelling equal terms with opposite sign.)

Note. *Area of triangle can easily be calculated with the help of following figure:*



Write the co-ordinates of the vertices as shown. Numbers to be multipled *as shown by arrows. All products with arrow downwards are of positive sign and are added to the products with arrow upwards with negative sign. Divide the sum by two.*

Cor.1. Area of a quadrilateral.

Area of a quadrilateral is the sum of the numerical values of areas of Δs formed by joining the two opposite vertices.

Cor. 2. Deduce the condition that the points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear.

If the points A, B, C are collinear i.e., there is no triangle formed by these points.

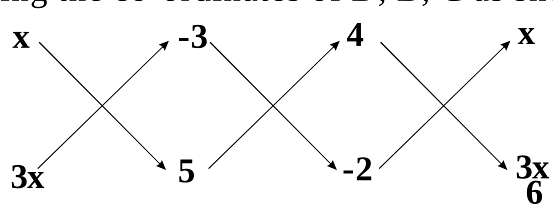
$$\text{Area of } \Delta ABC = 0$$

i.e., $(x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) = 0$

Example 6. The co-ordinates of A, B, C, D are (6,3), (-3,5) (4,-2) and (x, 3x) respectively such that

$$\frac{\Delta DBC}{\Delta ABC} = \frac{1}{2}\textbf{.}\ \text{Find x.}$$

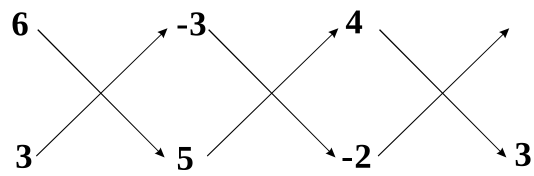
Solution. Writing the co-ordinates of D, B, C as shown



Area of $\Delta\, DBC$ = ½[5x+9x+6-20+12x+2x]

= ½[28x-14] ...(1)

Write the co-ordinates of A, B, C as shown



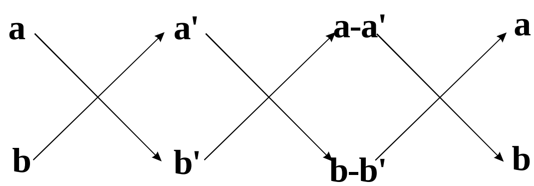
Area of ΔABC = ½[30+9+6-20+12+12]= 49/2 ...(2)

(1) and (2) give $\dfrac{\Delta DBC}{\Delta ABC} = \dfrac{28x-14}{49} = \dfrac{1}{2}$ (given)

It gives $8x-4 = 7$ or $x = 11/8$

Example 7. If the points (a,b), (a', b') and (a-a', b-b') be collinear, prove that ab'=a'b.

Solution. Writing the co-ordinates as shown



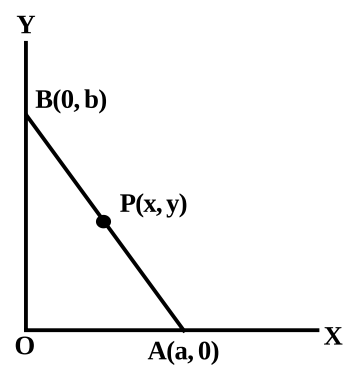
Area of Δ = ½[$(ab'-a'b)+a'(b-b')-b'(a-a') +(a-a')b-(b-b')a$]

$= ½(ab'-a'b)$

$\therefore$ Δ $= ½(ab'-a'b)=0$ as the points are collinear.

Hence $ab' = a'b$.

Example 8. If P(x,y) is any point on the line joining the points A(a,0) and B(0,b) then show that $\dfrac{x}{a} + \dfrac{y}{b} = 1$



Solution. As *P*, *A*, *B* are collinear

Area of ΔPAB can be put as zero and Area of ΔPAB

$$= \frac{1}{2}\begin{bmatrix} x & a & 0 & x \\ y & 0 & b & y \end{bmatrix}$$

$$\Rightarrow \ 0 = \frac{1}{2}[-ay+ab-bx]$$

$$\Rightarrow \ -ay+ab-bx = 0$$

$$\Rightarrow \ ab = ay+bx$$

Dividing by *ab*,

$$\frac{x}{a} + \frac{y}{b} = 1$$

5.5 LOCUS AND ITS EQUATION :

5.5.1 LOCUS OF A POINT

Locus or graph of a point is the path traced out by a moving point when it moves under some given conditions.

Illustration I. Let *P* be the moving point which moves in a plane keeping its distance same from the two fixed points *A* and *B* *i.e.*, *PA*=*PB*. Therefore, *P* traces a line through the middle point *D* of *AB* and ⊥ to *AB*. In this case locus is the right bisector of the line *AB*.

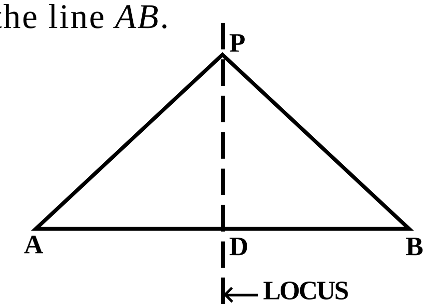
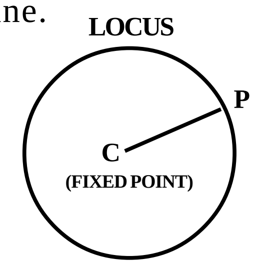


Illustration II. Let the point *P* move in plane, such that it keeps its distance same from a fixed point in the plane.



Moving point P can keep the distance same from a fixed point C in the plane if it moves on circle whose center is C and constant distance is the radius of the circle. In this case locus is a circle.

5.5.2 EQUATION OF LOCUS

As we have seen locus of a point is the curve or straight line traced out by moving point under some given conditions. By equation of locus, we mean that the equation in abscissa and ordinate which is satisfied by the co-ordinates of the every point on the locus(*i.e.*, every point traced out by moving point) and co-ordinates of no other point.

5.5.3 METHOD TO FIND THE EQUATION OF CURVE OR LOCUS OF MOVING POINT

Step 1. Let the co-ordinates of moving point be (x, y).

Step 2. Write down the given condition under which the point moves.

Step 3. Express the geometrical condition of step 2 in terms of x and y.

Step 4. Simplify, if required, the result in step 3. Equation so obtained will be the equation of the locus.

Example 9. Find the equation of the locus of a point which moves such that sum of its distance from two points (ae,0) and (-ae, 0) is 2a.

Solution. Let $P(x, y)$ be the moving point, such that distance of P from $(ae, 0)$+distance of P from $(-ae, 0) = 2a$.

$$\sqrt{(x-ae)^2 + y^2} + \sqrt{(x+ae)^2 + y^2} = 2a$$

Squaring,

$$\Rightarrow \quad (x+ae)^2+y^2 = 4a^2 + (x-ae)^2 + y^2 - 4a \sqrt{(x-ae)^2 + y^2}$$

$$\Rightarrow \quad x^2+a^2e^2+2aex+y^2 = 4a^2+x^2+a^2e^2-2aex+y^2-4a \sqrt{(x-ae)^2 + y^2}$$

$$\Rightarrow \quad \sqrt{4a \quad (x-ae)^2 + y^2} = 4a^2-4aex$$

$$\Rightarrow \quad \sqrt{(x-ae)^2 + y^2} = a-ex$$

Again Squaring,

$$\Rightarrow \quad x^2+a^2e^2-2aex+y^2 = a^2+e^2x^2-2aex$$

$$\Rightarrow \quad x^2+y^2 = a^2+e^2x^2-a^2e^2$$

$$\Rightarrow \quad x^2-e^2x^2 = a^2(1-e^2) -y^2$$

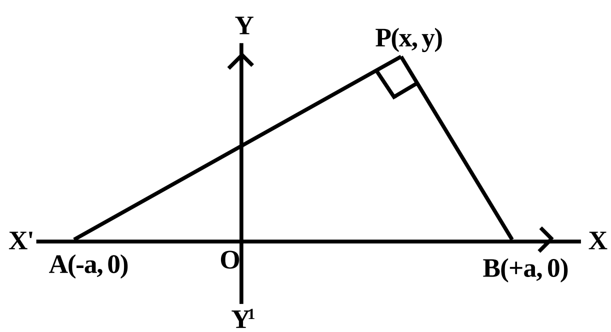
$$\Rightarrow \quad x^2(1-e^2)+y^2 = a^2(1-e^2)$$

$$\Rightarrow \quad x^2 + \frac{y^2}{1-e^2} = \frac{a^2(1-e^2)}{(1-e^2)} = a^2 \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$$

which is required equation of locus

Example 10. A and B are two fixed points. Find the locus of a point at which AB subtends a right angle.

Solution. Take the line joining the fixed points A, B as axis of x. Take middle point of AB as origin. Line through $O \perp AB$ should be taken as y-axis. Let $AO=OB=a$. Let $P(x,y)$ be moving point.



P moves such that $\quad APB = 90^1$

$\therefore \quad (AB)^2=(AP)^2+(PB)^2$

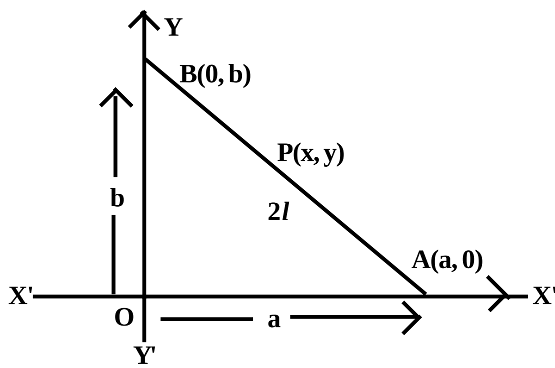
or $\quad (a+a)^2+(0-0)^2 = [(x+a)^2+y^2] + [(x-a)^2+y^2]$

or $\quad 4a^2 = 2(x^2+y^2+a^2)$

$\therefore$ Equation of locus is $x^2+y^2 = a^2$.

Example 11. A straight line of length 2l has its ends on the co-ordinate axes. Find the equation of locus of its middle point.

Solution. Let straight line AB have ends on the axes. Let $OA=a$ and $OB=b$. Co-ordinates of A and B are $(a,0)$ and $(0,b)$.



Let $P(x, y)$ be the co-ordinate of the mid-point. In all positions $AB= 2l$ or $AB^2=4l^2$ or $a^2+b^2=4l^2$ ...(1)

Mid point of AB is also given by $\left(\frac{a}{2}, \frac{b}{2}\right)$.

$\therefore \quad \frac{a}{2} = x \quad i.e., \quad a = 2x$

and $\quad \frac{b}{2} = y \quad i.e., \quad b = 2y$

and Substituting the values of a and b in (1)

Locus is $\quad 4x^2+4y^2 = 4l^2$

or $\quad x^2+y^2 = l^2$

5.6 STRAIGHT LINE :

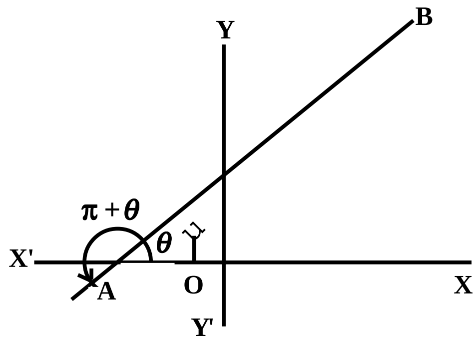
5.6.1 INTRODUCTION

A relation between x and y satisfied by all the points lying on the straight line and by no other point, is called the equation of that straight line. In order to find the equations of line in different forms, we will suppose a point $P(x, y)$ on the line and will write the condition under which P moves so that line is traced. Then we will find the locus of the point $P(x, y)$ which will the equation of the required line.

5.6.2 :- SLOPE OF A LINE

The slope of a line is the tangent of the angle which the part of the line above the x-axis makes with the positive direction of the x-axis (the angle measured is positive i.e., in counter-clock-wise).

Note. The slope of the line is independent of the sense of the line.



Consider sense AB, then slope = $\tan\theta$, and on considering sense BA, the slope = $\tan(\pi + \theta) = \tan\theta$.

The slope is the same for both senses of the line.

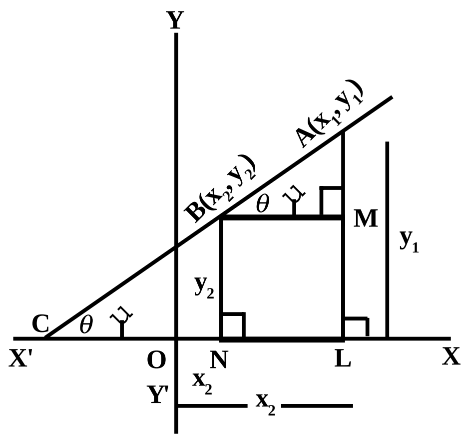
5.6.3 SLOPE OF STRAIGHT LINE JOINING TWO POINTS

To find the slope of the straight line joining two given points.

Let the straight line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ make an angle θ with the positive direction of the x-axis.

Draw AL, BN perpendiculars on the x-axis. Through B draw BM perpendicular on LA. Then $\underline{/MBA} = \underline{/XCA} = \theta$.

From right angled triangle AMB, we have



$$\tan\theta = \frac{MA}{BM} = \frac{LA - LM}{LN}$$

$$= \frac{LA - LM}{OL - ON}$$

$$= \frac{LA - BN}{OL - ON}$$

$$\therefore \quad m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1} \qquad (\because \tan\theta = m)$$

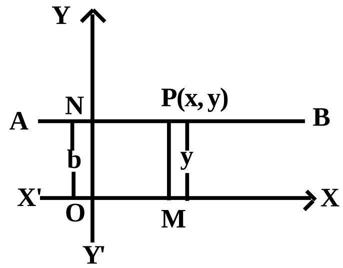
$$\text{Thus, slope} = \frac{\text{Difference of y co-ordinates}}{\text{Difference of x co-ordinates}},$$

difference taken in the same order.

5.6.4 STRAIGHT LINE PARALLEL TO THE X-AXIS.

To find the equation of a straight line parallel to the x-axis at a distance "b" from it. Hence, deduce the equation of x-axis.

Let AB be the line || to the x-axis meeting the y-axis in N such that $ON = b$.



Let $P(x, y)$ be any point on AB. Draw $PM \perp$ on the x-axis.

Then $MP = ON = b$

$y = b$, which is the required equation.

Deduction. x-axis is a line || to x-axis at a distance zero from it, hence its equation is $y = 0$.

Note. *If the line is above x-axis (as in figure), 'b' is positive and if it is below, 'b' is negative.*

5.6.5 STRIAGHT LINE PARALLEL TO THE Y-AXIS

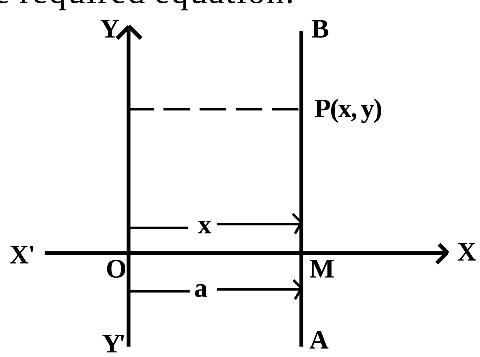
To find the equation of a straight line parallel to the y-axis at a distance "a" from it. Hence deduce the equation of y-axis.

Let AB be the line parallel to the y-axis. meeting the x-axis in M such that $OM = a$.

Let $P(x,y)$ be any point on AB.

Then $OM = a$

$\therefore$ $x = a$, which is the required equation.



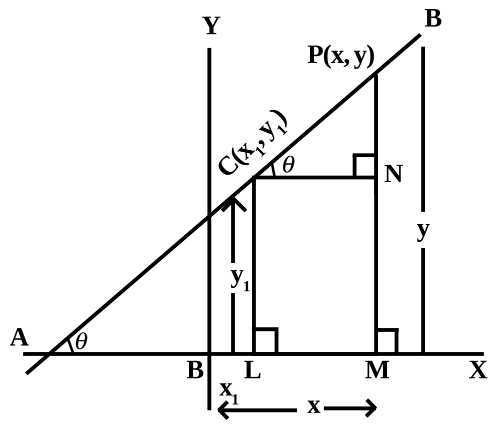
Deduction. *y*-axis is a line parallel to *y*-axis at a distance zero from it, Hence its equation is $x = 0$.

Note. *If the line is to the right of the y-axis, (as in the figure), 'a' is positive, and if it is to the left, 'a' is negative.*

5.6.6 EQUATION OF A LINE IN POINT-SLOPE FORM

To find the equation of a straight line drawn through a given point (x_1, y_1) in a given direction i.e., making a given angle θ with the x-axis.

Let the line AB pass through $C(x_1, y_1)$ and $\underline{/XAB} = \theta$. Then slope = $\tan\theta = m$ (say).



Let P(x, y) be the any point on AB.

Draw PM, CL perpendiculars on x-axis and CN is perpendicular to MP.

Clearly $\underline{/NCB} = \underline{/XAB} = \theta$

$$\therefore \quad m = \tan\theta = \frac{NP}{CN}$$

$$= \frac{MP - MN}{LM}$$

$$= \frac{MP - LC}{OM - OL}$$

$$m = \frac{y - y_1}{x - x_1}$$

i.e., $y - y_1 = m(x - x_1)$, which is the required equation of line *AB*.

Cor. If the line passes through the origin, then its equation becomes $y - 0 = m(x-0)$ *i.e.*, $y = mx$.

5.6.7 EQUATION OF A LINE IN TWO-POINT FORM

To find the equation of a (non-vertical) straight line passing through two given points (x_1, y_1) and (x_2, y_2).

The equation of any line passing through (x_1, y_1) is

$$y-y_1 = m(x-x_1) \qquad \text{(Point slope form)} \qquad ...(1)$$

Line (1) passes through (x_2, y_2), it will satisfy (1)

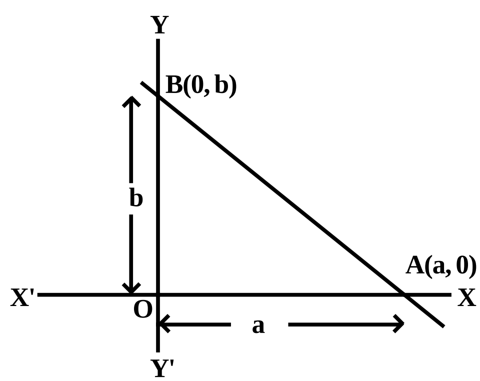
i.e., $$y_2-y_1 = m(x_2-x_1) \quad \text{or} \quad m = \frac{y_2 - y_1}{x_2 - x_1}$$

Substituting the value of *m* in (1), the equation of line is

$$y-y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

5.6.8 INTERCEPTS

Let a straight line (not passing through origin) *AB* meets the *x*-axis and *y*-axis in *A* and *B* repectively. Then



(*i*) Length of directed segment *OA* is called the **intercept cut by the line on *x*-axis or *x*-intercept.**

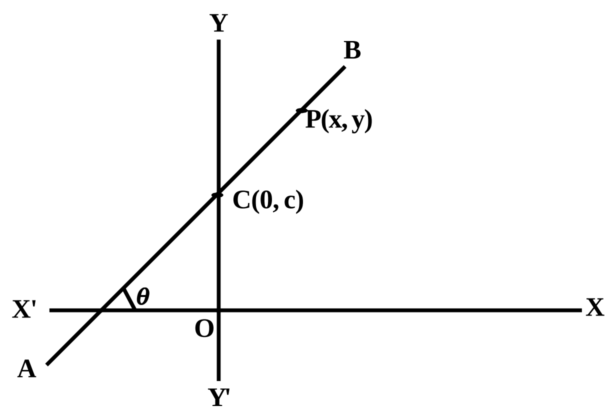
(*ii*) Length of directed segment *OB* is called the **intercept cut by the line on *y*-axis or *y*-intercept.**

(*iii*) *AB* is called the **portion of line intercepted between the axes.**

(*iv*) If $OA = a$, then *A* is $(a, 0)$; if $OB = b$, then $B\ (0, b)$.

5.6.9 EQUATION OF A LINE IN SLOPE-INTERCEPT FORM

"To find the equation of a straight line which cuts off a given intercept "*c*" on the *y*-axis and is inclined at a given angle θ to the *x*-axis."



Let a line makes an intercept '*c*' on the *y*-axis, having a slope $m = \tan\theta$. As the line makes intercept *c* on the *y*-axis, therefore it passes through $C(0, c)$. Using point slope, form equation of the line is

$$y - c = m(x - 0)$$

or $$y = mx + c.$$

Note. If the line passes through origin, then $c = 0$ and equation becomes $y = mx$.

Illustration. If a straight line makes an angle of 60º to the *x*-axis and cuts off an intercept 4 on the *y*-axis below the origin, then

$$m = \tan 60^\circ = \sqrt{3} \text{ and } c = -4$$

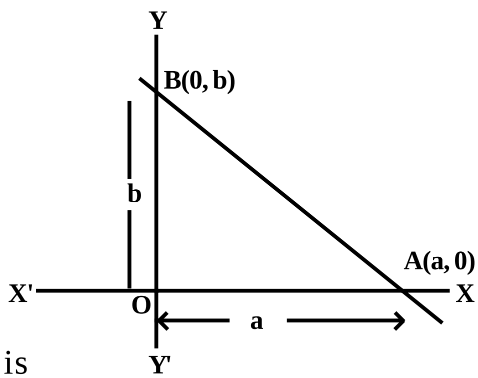
$$\therefore \text{ its equation is } y = \sqrt{3}\,x - 4.$$

5.6.10 EQUATION OF A LINE IN INTERCEPT FORM

To find the equation of a straight line which cuts off given intercepts 'a' and 'b' on the axes.

Let *AB* be the line meeting the axes in *A* and *B* such that $OA = a$ and $OB = b$.

Thus the co-ordinates of *A* and *B* are $(a, 0)$ and (o, b) repectively.



The equation of *AB* is

$$y - 0 = \frac{b-0}{0-a}(x-a)$$

or $$y = -\frac{b}{a}(x-a)$$

or $$\frac{y}{b} = -\frac{1}{a}(x-a)$$

$$= -\frac{x}{a} + 1$$

or $$\frac{x}{a} + \frac{y}{b} = 1$$, which is the required equation of line.

$$\left[\text{Remember : } \frac{x}{\text{Intercept on } x\text{-axis}} + \frac{y}{\text{Intercept on } y\text{-axis}} = 1.\right]$$

Cor I. If the line is parallel to the x-axis, then aDT , x/a D 0 and the equaion becomes $0 + \frac{y}{b} = 1$ or $y = b$.

Cor II. If the line is parallel to the y-axis, then bDT, y/b D 0 and the equation becomes $\frac{x}{a} + 0 = 1$ or $x = a$.

Illustration. Equation of the straight line making intercepts -4 and 6 on the axes is $\frac{x}{-4} + \frac{y}{6} = 1$ or $-3x + 2y = 12$.

Example 12. A straight line passes through the point (1, 1) and portion of the line intercepted between axes is divided by the point in the ratio 3 : 4. Find the equation.

Solution. AB is the portion of the line intercepted between axes. Line passes through (1, 1) and portion AB is divided at $P(1, 1)$ in the ratio 3 : 4.

Let the equation of the line in intercept form be $\frac{x}{a} + \frac{y}{b} = 1$. Clearly, co-ordinates of A and B are $(a, 0)$, $(0, b)$. Co-ordinates of P are $\left(\frac{4a}{7}, \frac{3b}{7}\right)$

$\therefore \quad \frac{4a}{7} = 1$ and $\frac{3b}{7} = 1$ which give $a = \frac{7}{4}$ and $b = \frac{7}{3}$

Hence, the equation of the line is

$$\frac{x}{7/4} + \frac{y}{7/3} = 1$$

or $$4x + 3y = 7$$

Example 13. Find the equation of the straight line passing through (-4, -5) and perpendicular to the straight line joining (1, 2) and (5, 6).

Solution. Slope of the line joining (1, 2) and (5, 6)

$$= \frac{6-2}{5-1} = \frac{4}{4} = 1.$$

$\therefore$ Slope of the line perpendicular to it = $-\frac{1}{1} = -1$

Equation of the required line is

$$y + 5 = -1(x+4) \qquad \text{| Slope Intercept Form |}$$

or $\quad y + 5 = -x - 4$

or $\quad x + y + 9 = 0.$

Example 14. Find the equation of a line which passes through (22, -6) and intercept on x-axis exceeds the intercept on y-axis by 5.

Solution. Let the line make an intercept *'a'* on *y*-axis.

$\therefore$ intercept on *x*-axis is 'a+5'

Equation of the line is $\frac{x}{a+5} + \frac{y}{a} = 1$ | Intercept Form |

It passes through (22, -6). $\therefore$ It will satisfy the equation

$$\therefore \qquad \frac{22}{a+5} - \frac{6}{a} = 1$$

Multiply both sides by $(a+5)\,a$

$$22a-6(a+5) = (a+5)a$$

or $\quad 22a-6a-30 = a^2+5a$

$\therefore \quad a^2-11a+30 = 0$

or $\quad (a-6)(a-5) = 0$

$\therefore \quad a=6,5$

If a = 6	If a = 5
Equation of line is	Equation of line is
$\frac{x}{11}+\frac{y}{6} = 1$	$\frac{x}{10}+\frac{y}{5} = 1$
or $6x+11y = 66$	or $x+2y = 10$

which are the required lines

5.6.11 GENERAL EQUATION OF A STRAIGHT LINE

To prove that any equation of the first degree in x and y represents a straight line.

Let the general equation of the first degree in x and y be

$$Ax + By + C = 0 \qquad ....(1)$$

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be any three points on the locus represented by (1).

Then the co-ordinates of these points satisfy (1)

$$Ax_1+By_1+C = 0 \qquad ....(2)$$

$$Ax_2+By_2+C = 0 \qquad ....(3)$$

$$Ax_3+By_3+C = 0 \qquad ....(4)$$

From (2) and (3), by substraction and transposition,

$$A(x_1-x_2) = -B(y_1-y_2) \qquad ....(5)$$

Similarly (3) and (4) give

$$A(x_2-x_3) = -B(y_2-y_3) \qquad ....(6)$$

Dividing (5) by (6), we have

$$\frac{x_1-x_2}{x_2-x_3} = \frac{y_1-y_2}{y_2-y_3}$$

After cross-multification and simplification, we get

$$x_1y_2-x_2y_1+x_2y_3-x_3y_2+x_3y_1-x_1y_3 = 0$$

or $\frac{1}{2}\left[(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)\right]=0$

i.e., Area of triangle formed by the three points is zero.

$\therefore$ The three points lies on a straight line. Therefore, locus is a straight line. Hence, the general equation of first degree in x and y represents a straight line.

5.7 SELF ASSESSMENT QUESTIONS :

1. Find the distance between following points

(i) (5,-12) and (9,-9)

(ii) (a,-a) & (-b, b)

2. Find the distance between the pair of points :

$(am_1^2, 2am_1)$ and $(am_2^2, 2am_2)$, where $m_1 > m_2$

3. Show tht the points (-2,3), (1,2) and (7,0) are collinear

4. Show that the points (3,3), (9,0), (12,21) are the vertices of the right angled triangle.

5. Two vertices of an equilateral triangle are (3,4) and (-2,3). Find the co-ordinates of the third vertex.

6. Show that the points (8,2), (5, -3) and (0,0) are the vertices of an isosceles triangle.

7. *A*, *B* are the points (3,4), (5,-2). Find the co-ordinates of point *P* such that $PA=PB$ and $\Delta PAB = 10$.

8. Points $(a,0)$, (o,b) and $(1,1)$ are collinear, show that

9. Prove that the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ are collinear.

10. Determine k in order that the points (2, -1), (-3,4) and $(k,5)$ may be collinear.

$$\frac{1}{a} + \frac{1}{b} = 1$$

11. Find the equation of locus of a point which moves so that distance from the axis of x is half its distance from the origin.

12. Find the locus of a point which moves so that the sum of squares of its distances from two fixed points is constant and equal to $2c^2$.

13. A point moves so that the difference of the squares of its distances from two fixed points $(a, 0)$ and $(-a, 0)$ is constant $= 2k^2$. Find the equation of the locus.

14. Two points $A(0, -2)$ and $B(0,2)$ are given. Find the locus of P such that $PA+PB = 7$.

15. Find the equation of the straight line passing through the points (1,2) and (0,5).

16. Find the equation of the line which passes through (4,5) and is parallel to the line $2x-3y-5 = 0$

17. Find the equation of a line passing through the point (2,1) and parallel to the line joining the points (1,3) and (-3,1)

18. Find the equation of the line passing through the points (4,5) and making intercepts on the axes which are equal in magnitude but opposite in sign.

5.8 KEY WORDS :

Co-ordinates, distance formula, area of triangle, locus of point, straight line, slope.

5.9 SUGGESTED READINGS :

1. Grewal, B.S. - Engineering Mathematics
2. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.
3. Ramana, B.V. - Higher Engineering Mathematics.
4. Aggarwal, R.S. - Modern Approach of Mathematics.

E E E

Subject	**:**	**Mathematics-I**	**Paper Code**	**:**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No.**	**:**	**06**
Lesson	**:**	**Differential Calculus**			

6.0 OBJECTIVES :

In this lesson, you will be able to understand

* the concept of limit of a function.
* the derivative and to write the derivatives of standard functions.
* differentiate functions using standard derivatives and rules of differentiation.
* higher order derivative of a function.
* successive differentation.
* Leibnitz's theorem.

6.1 INTRODUCTION :

Let 'f' be a function which is defined at every point in some open interval containing the point c, except possibly at the number 'c' itself and let 'l' be a real number. Then ' f ' is said to tend to the limit 'l' as x approaches 'c', if the difference between the value of f(x) and l can be made as small as we please by taking values of x very close to c & $x \neq c$, that is,

$$f(x) \to l \text{ as } x \to c$$

Definition : A function f(x) is said to tend to the limit 'l' as x approaches c, if corresponding to any given +ve number §, however small, we can find a +ve number δ, depending on ε, s.t.

$$| f(x) - l | < \varepsilon \text{ for } 0 < | x-c | < \delta$$

i.e., for every x in two intervals $c-\delta<x<c$ and $c<x<c+\delta$, $l-\varepsilon<f(x)<l+\varepsilon$.

In symbols, we write $\underset{x \to c}{\textbf{Lt}} f(x) = l$.

Right-handed limit : A function f(x) is said to tend to a limit l as $x \to c$ from the right (from above) if, to any positive number, §, however small, there corresponds a positive number δ, such that

$$| f(x) - l | < \varepsilon \text{ , for } c<x<c+\delta$$

In symbols, we write $\underset{x \to c+0}{\textbf{Lt}} f(x) = l$

<u>Left handed limit</u> : A function f(x) is said to tend to a limit l as $x \to c$ from the left (from below) if, to any positive number §, however small, there corresponds a positive number δ, such that

$$| f(x) - l | < \varepsilon \quad , \quad \text{for } c-\delta < x < c,$$

In symbols, we write $\underset{x \to c-0}{\textbf{Lt}} f(x) = l$

Thus, for the existence of a unique limit of f(x) as $x \to c$, the necessary and sufficient condition is

$$\underset{x \to c-0}{\textbf{Lt}} f(x) = \underset{x \to c+0}{\textbf{Lt}} f(x)$$

6.2 SOME STANDARD LIMITS :

(i) $\underset{x \to 0}{\textbf{Lt}} \dfrac{\sin x}{x} = 1$

(ii) $\underset{x \to 0}{\textbf{Lt}} \dfrac{\tan x}{x} = 1$

(iii) $\underset{n \to \infty}{\textbf{Lt}} \left(1 + \dfrac{1}{n}\right)^n = e$

(iv) $\underset{x \to 0}{\textbf{Lt}} [1+x]^{1/x} = e,$

(v) $\underset{x \to 0}{\textbf{Lt}} \dfrac{e^x - 1}{x} = 1$

Example 1. Evaluate $\underset{x \to 0}{\textbf{Lt}} \dfrac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Sol. Rationalising the given,

$$= \underset{x \to 0}{\mathbf{Lt}} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \times \frac{\sqrt{1+x} + \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \underset{x \to 0}{\mathbf{Lt}} \frac{(1+x) - (1-x)}{x[\sqrt{1+x} + \sqrt{1-x}]} = \underset{x \to 0}{\mathbf{Lt}} \frac{2}{\sqrt{1+x} + \sqrt{1-x}}$$

$$= \frac{2}{1+1} = 1$$

6.3 DIFFERENTIAL CO-EFFICIENT :

Let y=f(x) be a finite and single valued function of x, so that a small increment δx in x produces a small increment δy in the value of y, **then the** $\underset{\delta x \to 0}{\mathbf{Lt}} \frac{\delta y}{\delta x}$ **, if it exits,** is called **the differential co-efficient** or the derivative of f(x) w.r.t. x and is denoted by the symbol $\frac{dy}{dx}$, which is read as "dee y by dee x". The above *definition* can also be stated as :

The $\underset{\delta x \to 0}{\mathbf{Lt}} \frac{f(x+\delta x) - f(x)}{\delta x}$, if it exists is also called the differential co-efficient or derivative of f (x) w.r.t.x and is denoted by the symbol $\frac{d}{dx}$ f (x) or $f'(x)$ and read as f dash x.

Differentiation :-

The process of finding differential co-efficient of function is called "differentiation" and is generally denoted by the symbol $\frac{d}{dx}$. Thus, $\frac{d}{dx}(x^3)$ means that x^3 is to be differentiated with respect to x.

6.4 DIFFERENTIATION FROM FIRST PRINCIPLES OR AB INITIO OR FROM DEFINITION OR BY DELTA METHOD :

The process involves the following steps for differentiating a given function.

Step I. Put the given function equal to y.

i.e., Let $y = f(x)$.

Step II. Let δx be an increment of x and δy the corresponding increment of y, so that

$$y+\delta y = f(x+\delta x).$$

Step III. Find δy by subtracting y from $y+\delta y$.

i.e., $$\delta y = f(x+\delta x)-f(x).$$

Step IV. Obtain the quotient of the two increments

i.e., $$\frac{\delta y}{\delta x} = \frac{f(x+\delta x) - f(x)}{\delta x}$$

Step V. Find the limit of $\frac{\delta y}{\delta x}$ as $\delta x \to 0$.

This limit is the differential co-efficient of f(x) w.r.t. x.

When we differentiate a function by taking the above steps, the differentiation is said to be ab-initio or from first principles or from definition.

Example 2. *Find the differential co-efficient from definition of x^2.*

Sol. Let $$y = x^2 \quad ...(1)$$

Let δx be a small increment of x and δy the corresponding increment of y.

$\therefore$ $$y+\delta y = (x+\delta x)^2 \quad ...(2)$$

Subtracting (1) from (2),

$$\delta y = (x+\delta x)^2-x^2$$

$$= x^2+(\delta x)^2+2x\delta x-x^2$$

$$= (\delta x)^2+2x\delta x = \delta x\,(\delta x+2x)$$

Dividing both sides by δx,

$$\frac{\delta y}{\delta x} = \delta x+2x$$

Taking limits as $\delta x \to 0$.

$$\underset{\delta x \to 0}{\textbf{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\textbf{Lt}} (\delta x + 2x)$$

$$\frac{dy}{dx} = 2x.$$

Example 3. *Find the differential co-efficient from first principles of* $\sqrt{x}$

Sol. (i) Let $y = \sqrt{x}$

Let δx be a small increment of x and δy the corresponding increment of y, i.e., changing x to $x+\delta x$ and y to $y+\delta y$ in (1).

$$y+\delta y = \sqrt{x+\delta x} \qquad ...(2)$$

(2)-(1) gives, $\delta y = \sqrt{x+\delta x} - \sqrt{x}$

$$\delta y = \sqrt{x+\delta x} - \sqrt{x} \times \frac{\sqrt{x+\delta x} + \sqrt{x}}{\sqrt{x+\delta x} + \sqrt{x}}$$

[Rationalizing the num.]

$$= \frac{x+\delta x - x}{(\sqrt{x+\delta x} + \sqrt{x})} = \frac{\delta x}{\sqrt{x+\delta x} + \sqrt{x}}$$

Dividing both sides by δx, $\frac{\delta y}{\delta x} = \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$

Taking limits as $\delta x \to 0$

$$\underset{\delta x \to 0}{\textbf{Lt}} \frac{\delta y}{\delta x} = \underset{\delta x \to 0}{\textbf{Lt}} \frac{1}{\sqrt{x+\delta x} + \sqrt{x}}$$

Put $\delta x = 0$ in R.H.S.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

6.4.1 Differential Co-efficient of x^n, where n is real

Proof. Let $y = x^n$

Let δx be an increment of x and δy the corresponding increment of y.

< $$y+\delta y = (x+\delta x)^n$$

< $$\delta y = (x+\delta x)^n - x^n = x^n \left[\left(1 + \frac{\delta x}{x}\right)^n - 1\right]$$

As $\delta x \rightarrow 0$, $\frac{\delta x}{x} < 1$ in magnitude

< $\left(1 + \frac{\delta x}{x}\right)^n$ can be expanded by the Binomial Theorem for any index, whatever n may be.

< $$\delta y = x^n \left[1 + n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots \text{ terms containing higher powers of } \delta x - 1\right]$$

$$= x^n \left[n\left(\frac{\delta x}{x}\right) + \frac{n(n-1)}{2!}\left(\frac{\delta x}{x}\right)^2 + \ldots \text{ terms containing higher powers of } \delta x\right]$$

Dividing every term by δx

< $$\frac{\delta y}{\delta x} = x^n \left[n \cdot \frac{1}{x} + \frac{n(n-1)}{2!} \cdot \frac{\delta x}{x^2} + \ldots \text{ terms containing higher powers of } \delta x\right]$$

Proceeding to the limit as $\delta x \rightarrow 0$.

$$\underset{\delta x \rightarrow 0}{\mathbf{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx} = x^n \cdot \frac{n}{x} = n \cdot x^{n-1}.$$

Hence $$\frac{\mathbf{d}}{\mathbf{dx}}(x^n) . = n \cdot x^{n-1}.$$

6.4.2 Differential Co-efficient of $(ax+b)^n$, where n is real

Let $\quad y = (ax+b)^n.$

Let δx be an increment of x and δy the corresponding increment of y.

‹ $$y+\delta y = [a(x+\delta x)+b]^n$$

$$\delta y = [(ax+b)+a\delta x]^n-(ax+b)^n$$

$$= (ax+b)^n \left[\left(1+\frac{a\delta x}{ax+b}\right)^n - 1\right]$$

As $\delta x \to 0$, $\dfrac{a\delta x}{ax+b} < 1$ in magnitude.

‹ $\left(1+\dfrac{a\delta x}{ax+b}\right)^n$ can be expanded by the Binomial Theorem for any index, whatever 'n' may be.

‹ $$\delta y = (ax+b)^n \left[1+n\left(\frac{a\delta x}{ax+b}\right)+\frac{n(n-1)}{2!}\left(\frac{a\delta x}{ax+b}\right)^2 \delta x + \ldots\right.$$

$$\left. \ldots\ldots\text{terms containing higher powers of } \delta x - 1\right]$$

Dividing every term by δx

‹ $$\frac{\delta y}{\delta x} = (ax+b)^n \left[n\left(\frac{a}{ax+b}\right)+\frac{n(n-1)}{2!}\left(\frac{a}{ax+b}\right)^2 \delta x + \ldots\right.$$

$$\left. \ldots\ldots\text{terms containing higher powers of } \delta x\right]$$

Taking Limit as $\delta x \to 0$,

$$\underset{\delta x \to 0}{\textbf{Lt}} \frac{\delta y}{\delta x} = \frac{dy}{dx} = (ax+b)^n . \frac{na}{ax+b}$$

$$= n(ax+b)^{n-1} a.$$

Hence $\boldsymbol{\dfrac{d}{dx}(ax+b)^n = n(ax+b)^{n-1}\, a.}$

Some useful results :

1. Derivative of a constant is zero, that is, $\frac{d}{dx}(c) = 0$, Where c is a Constant

II. The Derivative of the product of a constant and a derivable function is the product of the constant and the derivative of the function, that is

$$\frac{d}{dx}(cu) = c.\frac{d}{dx}u$$

III. The derivative of the algebric sum or difference of a finite number of derivable functions is the algebric sum or difference of their derivatives, that is,

Let $y = u+v-w+.....$, where u, v, w,... are functions of x, then

$$\frac{d}{dx}(u+v-w+...) = \frac{du}{dx} + \frac{dv}{dx} - \frac{dw}{dx}$$

Example 4. *Differentiate the following functions w.r.t. x :*

(i) $ax^4+bx^3+cx^2-dx-e$

(ii) $\left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$

Sol. (i) Let $y = ax^4+bx^3+cx^2-dx-e$

$$\therefore \quad \frac{dy}{dx} = \frac{d}{dx}(ax^4)+\frac{d}{dx}(bx^3)+\frac{d}{dx}(cx^2) - \frac{d}{dx}(d\,x) - \frac{d}{dx}(e)$$

$$= 4ax^3+3bx^2+2cx-d$$

(ii) Let $y = \left(x - \frac{1}{x}\right)\left(x^2 - \frac{1}{x^2}\right)$

$$= x^3 - \frac{1}{x} - x + \frac{1}{x^3}$$

$$= x^3 - x^{-1} - x + x^{-3}$$

$$\frac{dy}{dx} = 3x^2+x^{-2}-1-3x^{-4} \quad = \quad 3x^2 + \frac{1}{x^2} - 1 - \frac{3}{x^4}$$

6.5 DERIVATIVE OF THE PRODUCT OF TWO FUNCTIONS :

The derivative of the product of two derivable functions is equal to the sum of the products of each function multiplied by the derivative of the other i.e.,

$$\mathbf{\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}}$$

where u and v are functions of x

Example 5. *Differentiate the following* :

(i) $(2x+3)(3-4x)$

(ii) $(3x+4)^2 (2x-3)^3$

Sol. (i) Let $y = (2x+3)(3-4x)$

‹ $$\frac{dy}{dx} = (2x+3)\frac{d}{dx}(3-4x) + (3-4x)\frac{d}{dx}(2x+3)$$

$$= (2x+3)(-4)+(3-4x)(2)$$

$$= -8x-12+6-8x = -(16x+6)$$

(ii) Let $y = (3x+4)^2 (2x-3)^3$

‹ $$\frac{dy}{dx} = (3x+4)^2\frac{d}{dx}(2x-3)^3 + (2x-3)^3\frac{d}{dx}(3x+4)^2$$

$$= (3x+4)^2 [3(2x-3)^2 \; 2] + (2x-3)^3 [2(3x+4) \; 3]$$

$$= 6(3x+4)(2x-3)^2 [(3x+4) + (2x-3)]$$

$$= 6(3x+4)(2x-3)^2 (5x+1)$$

6.6 DERIVATIVE OF A QUOTIENT :

If $y = \frac{u}{v}$, where u and v are functions of x, prove that

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Remember : The differential co-efficient of the quotient of two functions is equal to

$$\frac{\text{Denom.}\times[\text{diff. co-eff of Num.}]-\text{Num.}\times(\text{diff. co-eff. of the denom.})}{(\text{Denom})^2}$$

Example 6. *Differentiate* $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$

Sol. Let $y = \frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$

$$\frac{dy}{dx} = \frac{(\sqrt{a}-\sqrt{x})\frac{d}{dx}(\sqrt{a}+\sqrt{x}) - (\sqrt{a}+\sqrt{x})\frac{d}{dx}(\sqrt{a}-\sqrt{x})}{(\sqrt{a}-\sqrt{x})^2}$$

$$= \frac{(\sqrt{a}-\sqrt{x})\frac{1}{2\sqrt{x}} - (\sqrt{a}+\sqrt{x})\left(\frac{-1}{2\sqrt{x}}\right)}{(\sqrt{a}-\sqrt{x})^2}$$

$$= \frac{\frac{(\sqrt{a}-\sqrt{x})}{2\sqrt{x}} + \frac{(\sqrt{a}+\sqrt{x})}{2\sqrt{x}}}{(\sqrt{a}-\sqrt{x})^2}$$

$$= \frac{\sqrt{a}-\sqrt{x}+\sqrt{a}+\sqrt{x}}{2\sqrt{x}(\sqrt{a}-\sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{2\sqrt{a}}{2\sqrt{x}(\sqrt{a} - \sqrt{x})^2}$$

$$\frac{dy}{dx} = \frac{\sqrt{a}}{\sqrt{x}(\sqrt{a} - \sqrt{x})^2}$$

6.7 FUNCTION OF A FUNCTION, *i.e.*, COMPOSITE FUNCTION :

Consider the relations $y = u^2+2u+6$ and $u = x^3+4x+3$. Evidently y is a function of u and, u in turn, is a function of x. So y is a function of a function x.

Hence if y is a function of u, and u, in turn is a function of x, we say that y is a function of x.

Theorem Derivative of a Function of a Function *i.e.* derivative of composite functions

if $y = f(u)$, is a function of u,

and $u = \phi(x)$ is a function of x,

then $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Cor. The above result can be generalised, that is,

If $y = f_1(x_1)$, $x_1 = f_2(x_2)$,

$x_2 = f_3(x_3)$,......, $x_n = f_{n+1}(x)$

Then $$\frac{dy}{dx} = \frac{dy}{dx_1} \cdot \frac{dx_1}{dx_2} \cdot \frac{dx_2}{dx_3} \cdot .... \frac{dx_n}{dx}$$

which is called the **Chain Rule.**

Example 7. *Differentiate :*

(i) $\dfrac{1}{(ax^2+2bx+c)^n}$

(ii) $\sqrt{\dfrac{1-x}{1+x}}$

Sol. Let $y = \dfrac{1}{(ax^2+2bx+c)^n}$

Put $ax^2+2bx+c = u$

< $y = \dfrac{1}{u^n} = u^{-n}$

< $\dfrac{dy}{du} = -n u^{-n-1}$; and $\dfrac{du}{dx} = 2ax+2b$; and

Hence $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$= -n.u^{-(n+1)}(2ax+2b) = -\dfrac{2n(ax+b)}{(ax^2+2bx+c)^{n+1}}$

or directly.

Let $y = \dfrac{1}{(ax^2+2bx+c)^n} = (ax^2+2bx+c)^{-n}$

$\dfrac{dy}{dx} = -n(ax^2+2bx+c)^{-n-1}.\dfrac{d}{dx}(ax^2+2bx+c)$

$= -n(ax^2+2bx+c)^{-(n+1)}.(2ax+2b)$

$= -\dfrac{2n(ax+b)}{(ax^2+2bx+c)^{n+1}}$

(ii) Let $y = \sqrt{\dfrac{1-x}{1+x}}$

Put $u = \dfrac{1-x}{1+x}$ so that $y = u^{-½}$

< $$\frac{dy}{du} = \frac{1}{2} u^{-1/2} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-½}$$

and $$\frac{du}{dx} = \frac{(1+x) \cdot \frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= \frac{(1+x)(-1) - (1-x) \cdot 1}{(1+x)^2} = -\frac{2}{(1+x)^2}$$

Hence $$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2} \frac{-2}{(1+x)^2} = \frac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$$

or direct'y

Let $$y = \sqrt{\frac{1-x}{1+x}} = \left(\frac{1-x}{1+x}\right)^{1/2}$$

< $$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{1/2} \frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

| < $\dfrac{d}{dx} u^n = n u^{n-1} \dfrac{du}{dx}$

< $$\frac{dy}{dx} = \frac{1 \cdot 1}{2\left(\frac{1-x}{1+x}\right)^{1/2}} \left[\frac{(1+x) \cdot \frac{d}{dx}(1-x) - (1-x)\frac{d}{dx}(1+x)}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{(1+x)^{1/2}}{(1-x)^{1/2}} \left[\frac{(1+x)(-1) - (1-x) . 1}{(1+x)^2}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\frac{-2}{(1-x)^{1/2}(1+x)^{3/2}} = \frac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$$

Example 8. *Differentiate w.r.t.* x

$$\sqrt{\frac{x^2-2ax}{a^2-2ab}}$$

Sol. Let $y = \sqrt{\frac{x^2-2ax}{a^2-2ab}} = \frac{1}{\sqrt{a^2-2ab}} (x^2-2ax)^{1/2}$

Then $\frac{dy}{dx} = \frac{1}{\sqrt{a^2-2ab}} \cdot \frac{d}{dx}(x^2-2ax)^{1/2}$

$$= \frac{1}{\sqrt{a^2-2ab}} \cdot \frac{1}{2}(x^2-2ax)^{-1/2} \frac{d}{dx}(x^2-2ax)$$

$$= \frac{1}{\sqrt{a^2-2ab}} \cdot \frac{1}{2}(x^2-2ax)^{-1/2} (2x-2a)$$

$$\frac{dy}{dx} = \frac{x-a}{\sqrt{a^2-2ab}.\sqrt{x^2-2ax}}$$

Example 9. *if* $x\sqrt{1+y} + y\sqrt{1+x} = 0,$

prove that $\frac{dy}{dx} = \frac{-1}{(1+x^2)}$

Sol. $x\sqrt{1+y} + y\sqrt{1+x} = 0$

or $x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides,

$$x^2(1+y) = y^2(1+x)$$

or $\quad x^2+x^2y = y^2+y^2x$

or $\quad x^2-y^2 = -x^2y+y^2x$

or $\quad (x-y)(x+y) = -xy(x-y)$

Dividing both sides by x-y,

$$x+y = -xy$$

or $\quad y+xy = -x$ or $y(1+x) = -x$

‹ $\quad y = -\dfrac{x}{1+x}$

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}(x) - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$\frac{dy}{dx} = -\left[\frac{1+x-x}{(1+x)^2}\right] = \frac{-1}{(1+x)^2}.$$

6.8 DIFFERENTIATION OF PARAMETRIC EQUATIONS :

Theorem. **If x and y be expressed in terms of any variable parameter t, then**

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Example 10. *if* $x = \dfrac{a(1-t^2)}{1+t^2}$ *and* $y = \dfrac{2bt}{1+t^2}$, *find* $\dfrac{dy}{dx}$.

Sol. $\quad x = a\left[\dfrac{1-t^2}{1+t^2}\right]$

$$\frac{dx}{dt} = a\left[\frac{(1+t^2)(-2t) - (1-t^2)\,2t}{(1+t^2)^2}\right] = \frac{-4at}{(1+t^2)^2}$$

Also $\quad y = \dfrac{2bt}{1+t^2}$

$$= 2b\left[\frac{(1+t^2).1 - t.\,2t}{(1+t^2)^2}\right] = 2b\left[\frac{(1-t^2)}{(1+t^2)^2}\right]$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2b(1-t^2)}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-4at} = -b\,\frac{(1-t^2)}{2at}\ .$$

6.9 DIFFERENTIATION OF IMPLICIT FUNCTIONS :

To find the value of $\frac{dy}{dx}$ when y is given as an implicit function of x, we differentite both sides of the equation w.r.t. x regarding y as a function of x and then solve the resulting equation for $\frac{dy}{dx}$.

Example 11. *If* $ax^2+2hxy+by^2+2gx+2fy+c = 0$, find $\frac{dy}{dx}$.

Sol. $ax^2+2hxy+by^2+2gx+2fy+c = 0$

Differentiating w.r.t. x, we have

$$a\,.\frac{d}{dx}(x^2) + 2h\,.\frac{d}{dx}(xy) + b.\frac{d}{dx}(y^2) + 2g.\frac{d}{dx}(x) + 2f\,.\frac{d}{dx}(y) + \frac{d}{dx}(c) = 0$$

or $$2ax + 2h\left[x\,.\frac{dy}{dx} + y\,.\,1\right] + 2b\,.\,y\frac{dy}{dx} + 2g + 2f\,\frac{dy}{dx} = 0$$

or $$2ax + 2hx\frac{dy}{dx} + 2hy + 2by\frac{dy}{dx} + 2g + 2f\frac{dy}{dx} = 0$$

or $$2\frac{dy}{dx}[hx+by+f] = -2[ax+hy+g]$$

‹ $$\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}$$

6.10 DIFFERENTIATION OF LOGARITHMIC FUNCTIONS :

Definition of Logarithm

If $a^x = N$, then x is called the logarithm of N to the base a, i.e., the logarithm of a number to a given base is the power to which the base must be raised to equal that number. The logarithm of N to base a is denoted by $\log_a N$. Logarithms to the base 10 are called Common Logarithms. They are used in numerical calculations. Logarithms to the base e are called Natural Logarithms.

In Calculus and in higher mathematics, the number

$$e = \underset{n\to\infty}{\textbf{Lt}} \left(1 + \frac{1}{n}\right)^n$$

is taken as the base. So here in Caluclus we shall not express any base, the base 'e' will always be understood.

The following elementary formulae of logarithms and limits with which the student is already familiar should be comitted to memory :

(i) $\log_a (mn) = \log_a m + \log_a n.$

(ii) $\log_a m/n = \log_a m - \log_a n.$

(iii) $\log_a m^n = n \log_a m.$

(iv) $\log_a m = \log_b m \times \log_a b.$

(v) $\log_b a \times \log_a b = 1$ or $\log_b a = \dfrac{1}{\log_a b}$

(vi) $\log_a 1 = 0$

(vii) $\log_a a = 1$

(viii) $\underset{x\to 0}{\textbf{Lt}} \dfrac{a^x-1}{x} = \log a$

(ix) $e^{\log f(x)} = f(x)$

6.10.1 Differential Co-efficient of $\log_a x$

Let $y = \log_a x$

If δx is the increment in x and δy is the coresponding increment in y, then

$$y + \delta y = \log_a (x+\delta x)$$

$$\delta y = \log_a (x+\delta x) - \log_a x$$

$$= \log_a \frac{x+\delta x}{x} = \log_a \left(1+\frac{\delta x}{x}\right)$$

Dividing by δx, we have $\frac{\delta y}{\delta x} = \frac{1}{\delta x} . \log_a \left(1+\frac{\delta x}{x}\right)$

Dividing & multiplying by x on R.H.S.,

$$= \frac{1}{x} . \frac{x}{\delta x} \log_a \left(1+\frac{\delta x}{x}\right)$$

$$= \frac{1}{x} . \log_a \left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} \qquad \text{[Note]}$$

As $\delta x \to 0$. $\underset{\delta x \to 0}{\text{Lt}} \left(1+\frac{\delta x}{x}\right)^{\frac{x}{\delta x}} = e$

$$\left(\because \underset{x \to 0}{\text{Lt}} (1+x)^{1/x} = e\right)$$

Proceeding to the limit, when $\delta x \to 0$, we have

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} . \frac{\delta y}{\delta x} = \frac{1}{x} \log_a e$$

Hence, $\frac{d}{dx} (\log_a x) = \frac{1}{x} \log_a e$

Cor. 1. Differentiation of log x.

Put a = e in above so that $y = \log_e x = \log x$

$$\frac{dy}{dx} = \frac{1}{x} \cdot \log_e e = \frac{1}{x} \times 1 = \frac{1}{x} \qquad [‹ \ \log_a a = 1]$$

Hence, $\frac{dy}{dx}(\log_e x) = \frac{1}{x}$.

Similarly, if $y = \log u$

then $\frac{dy}{du} = \frac{1}{u}$

and $\frac{d}{dx}(\log u) = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{u} \times \frac{du}{dx}$

Remember that

$$\frac{d}{dx}\log(\text{function}) = \frac{1}{\text{function}} \times \frac{d}{dx}(\text{function})$$

e.g., $[\log(3x+4)] = \frac{1}{3x+4} \cdot \frac{d}{dx}(3x+4) = \frac{3}{3x+4}$

Example 12. *Differentiate :*

(i) $y = \log \sqrt[7]{x^3}$

(ii) $y = \log(x^2+3x+5)^{1/5}$

Sol. (i) $y = \log x^{3/7} = \frac{3}{7} \log x$

‹ $\frac{dy}{dx} = \frac{3}{7} \cdot \frac{d}{dx}(\log x) = \frac{3}{7} \cdot \frac{1}{x}$

(ii) $y = \log(x^2+3x+5)^{1/5}$

‹ $= \frac{1}{5}\log(x^2+3x+5)$

Put $u = x^2+3x+5$

‹ $y = \frac{1}{5}\log u$

‹ $\frac{dy}{du} = \frac{1}{5} \cdot \frac{1}{u}$ and $\frac{du}{dx} = 2x+3$

<

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{5} \cdot \frac{1}{u} \times (2x+3) = \frac{(2x+3)}{5(x^2+3x+5)}$$

Or Directly

Let $y = \log (x^2+3x+5)^{1/5}$

$$= \frac{1}{5} \log (x^2+3x+5)$$

$$= \frac{1}{5} \frac{d}{dx} \log (x^2+3x+5)$$

$$= \frac{1}{5} \frac{1}{(x^2+3x+5)} \frac{d}{dx} (x^2+3x+5)$$

$$\therefore \quad \log u = \frac{1}{u} \frac{d}{dx} (u).$$

$$= \frac{1}{5} \frac{1}{(x^2+3x+5)} (2x+3).$$

6.11 RESULTS ON DIFFERENTIATION OF TRIGOMETRIC & EXPONENTIAL FUNCTIONS :

(i) $\frac{d}{dx} (\sin x) = \cos x$

(ii) $\frac{d}{dx} (\cos x) = -\sin x$

(iii) $\frac{d}{dx} (\tan x) = \sec^2 x$

(iv) $\frac{d}{dx} (\cot x) = -\text{cosec}^2 x$

(v) $\frac{d}{dx} (\sec x) = \sec x . \tan x$

(vi) $\frac{d}{dx} (\text{cosec } x) = -\text{cosec } x . \cot x$

(vii) $\frac{d}{dx}(a^x) = a^x \log a$

(viii) $\frac{d}{dx}(a^u) = a^u \log a \cdot \frac{du}{dx}$

(ix) $\frac{d}{dx}(e^x) = e^x$

(x) $\frac{d}{dx}(e^u) = e^u \frac{du}{dx}$

(xi) $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$

(xii) $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$

(xiii) $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x \cdot \sqrt{x^2-1}}$

6.12 SUCCESSIVE DIFFERENTIATION :

If y be a function of x, then $\frac{dy}{dx}$ is called the first differential co-efficient or first derivative of y w.r.t. x. If this derivative is again differential function then its derivative i.e., $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ is called second differential co-efficient or second derivative of y w.r.t. x, which we denote as $\frac{d^2y}{dx^2}$. In like manner, the third differential co-efficient or third derivative of y w.r.t. x means the differential co-efficient of $\frac{d^2y}{dx^2}$ i.e., $\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and is represented by $\frac{d^3y}{dx^3}$ and so on. In general, the nth differential co-efficient of y w.r.t. x is denoted by $\frac{d^ny}{dx^n}$. This process of finding the differential co-efficient of a function is called **Successive Differentiation.**

Thus, if y = f(x), the succesive differential co-efficients of f(x) are $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3} \ldots\ldots \frac{d^ny}{dx^n}$.

These are also denoted by :

(i) $y_1, y_2, y_3, ..., y_n$

(ii) $y', y'', y''', ..., y^n$

(iii) $Dy, D^2y, D^3y, ..., D^ny$.

(iv) $f(x), f'(x), f''(x), ..., f^n(x)$

Example 13. Find $\frac{d^3y}{dx^3}$, when $y = 4x^3+4x+2$.

Sol. Here, $y = 4x^3+4x+2$

‹ $\frac{dy}{dx} = 12x^2+4$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(12x^2+4) = 24x$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}(24x) = 24$$

Example 14. If $y = \frac{x}{\sqrt{1-x^2}}$, find $\frac{d^3y}{dx^3}$.

Sol. $y = \frac{x}{\sqrt{1-x^2}}$

‹ $\frac{dy}{dx} = \frac{\sqrt{1-x^2}.1 - x.\frac{1}{2}(1-x^2)^{-1/2} \times (-2x)}{(1-x^2)}$

$$= \frac{\sqrt{1-x^2} + \frac{x^2}{\sqrt{1-x^2}}}{(1-x^2)} = \frac{1-x^2+x^2}{(1-x^2)^{3/2}} = (1-x^2)^{-3/2}$$

$$\frac{d^2y}{dx^2} = -\frac{3}{2}(1-x^2)^{-5/2} \times (-2x) = 3x(1-x^2)^{-5/2}$$

$$\frac{d^3y}{dx^3} = 3\left[x\left(-\frac{5}{2}\right)(1-x^2)^{-7/2} \times (-2x) + (1-x^2)^{-5/2}\right]$$

$$= 15\, x^2(1-x^2)^{-7/2} \times 3(1-x^2)^{-5/2}$$

$$= \frac{15x^2}{(1-x^2)^{7/2}} + \frac{3}{(1-x^2)^{5/2}} = \frac{3(1-x^2) + 15x^2}{(1-x^2)^{7/2}}$$

$$\frac{d^3y}{dx^3} = \frac{3+12x^2}{(1-x^2)^{7/2}} = \frac{3(1+4x^2)}{(1-x^2)^{7/2}}$$

Example 15. If $y = \sin (a \sin^{-1} x)$, prove that

$$(1-x^2) \frac{d^2y}{dx^2} = x \frac{dy}{dx} - a^2y.$$

Sol. $y = \sin (a \sin^{-1} x)$ .....(1)

$$\therefore \frac{dy}{dx} = \cos (a \sin^{-1} x) \frac{a}{\sqrt{1-x^2}} = \frac{a . \cos (a \sin^{-1} x)}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2} \frac{dy}{dx} = a \cos (a \sin^{-1} x)$$

Squaring both sides,

$$(1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 \cos^2 (a \sin^{-1} x)$$

or $$(1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 [1 - \sin^2 (a \sin^{-1} x)]$$

(using $\cos^2 \theta = 1-\sin^2\theta$)

or $$(1-x^2) \left(\frac{dy}{dx}\right)^2 = a^2 (1-y^2)$$ [By (1)]

Again differentiating w.r.t. x, we have

$$(1-x^2) \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)^2 \frac{d}{dx} (1-x^2) = a^2 \left[0 - \frac{d}{dx} y^2\right]$$

or $$(1-x^2)\, 2 \frac{dy}{dx} \frac{d^2y}{dx^2} - 2x \left(\frac{dy}{dx}\right)^2 = -2a^2y \frac{dy}{dx}$$

Dividing every term by $2\frac{dy}{dx}$,

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -a^2y$$

or $$(1-x^2)\frac{d^2y}{dx^2} = x\frac{dy}{dx} - a^2y.$$

Example 16. If x = a (cos θ+θ sinθ), y = a (sin θ-θ cosθ), prove that

$$a\theta\frac{d^2y}{dx^2} = \sec^3\theta.$$

Sol. We have, $$\frac{dx}{d\theta} = a(-\sin\theta + \theta\cos\theta + \sin\theta) = a\theta\cos\theta$$

$$\frac{dy}{d\theta} = a(\cos\theta + \theta\sin\theta - \cos\theta) = a\theta\sin\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx} = \frac{a\theta\sin\theta}{a\theta\cos\theta} = \tan\theta$$

Again, differentiating both sides w.r.t. x

$$\frac{d^2y}{dx^2} = \sec^2\theta\cdot\frac{d\theta}{dx} = \sec^2\theta\cdot\frac{1}{a\theta\cos\theta} = \frac{\sec^3\theta}{a\theta}$$

Hence $$a\theta\frac{d^2y}{dx^2} = \sec^3\theta.$$

Example 17. If $y = e^{ax}\sin bx$, prove that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0$$

Solution : We have $y = e^{ax}\sin bx$ ...(1)

$$\frac{dy}{dx} = be^{ax}\cos bx + a e^{ax}\sin bx$$

or $$= be^{ax}\cos bx + ay$$ [by (1)] ...(2)

< $$\frac{d^2y}{dx^2} = -b^2 e^{ax} \sin bx + ab\, e^{ax} \cos bx + a \frac{dy}{dx}$$

$$= -b^2y + a\left[\frac{dy}{dx} - ay\right] + a\frac{dy}{dx} \qquad \text{[by (1) \& (2)]}$$

< $$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)\, y = 0$$

Example 18. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

Sol. We have,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(1)$$

< $$\frac{d}{dx}\left(\frac{x^2}{a^2}\right) + \frac{d}{dx}\frac{y^2}{b^2} = \frac{d}{dx}(1)$$

or $$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

< $$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

or $$\frac{dy}{dx} = -\frac{b^2x}{a^2y} \qquad ...(2)$$

Again differentiating both sides w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{-b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right) = -\frac{b^2}{a^2}\left[\frac{y\,.\,1 - x\frac{dy}{dx}}{y^2}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{y + \frac{b^2x^2}{a^2y}}{y^2}\right] \qquad \text{[by (2)].}$$

$$= -\frac{b^2(a^2y^2+b^2x^2)}{a^4y^3} = \frac{-b^2 . a^2b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}{a^4y^3}$$

$$\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3} \qquad \text{[by (1)].}$$

6.13 DERIVATIVES OF THE NTH ORDER OF SOME STANDARD FUNCTIONS OF X :

6.13.1 To find the nth differential of x^m.

Let $y = x^m$

then $y_1 = mx^{m-1}$

$y_2 = m(m-1)x^{m-2}$

$y_3 = m(m-1)(m-2)\ x^{m-3}$

....

....

‹ $y_n = [m(m-1)(m-2).......\text{upto n factors}] \times x^{m-n}$

$y_n = m(m-1)(m-2).......(m-n+1)\ .\ x^{m-n}$

where $n < m$.

Cor. If m be a positive integer,and if n = m

then $y_m = m(m-1)(m-2).......(m-m+1)\ x^{m-m}$

$= m(m-1)(m-2).......3.\ 2.\ 1 = \underline{|m}$

i.e., $\frac{d^m}{dx^m}(x^m) = \underline{|m}$

6.13.2 The nth differential co-efficient of $(ax+b)^m$, where $n<m$.

Let $y = (ax+b)^m$

then $y_1 = m(ax+b)^{m-1}.\ a$

$y_2 = m(m-1)(ax+b)^{m-2}.\ a^2$

$$y_3 = m(m-1)(m-2)(ax+b)^{m-3} . a^3$$

....

....

$$\therefore \quad y_n = m(m-1)(m-2)(m-3).......(m-n+1) \times (ax+b)^{m-n} . a^n$$

Cor. If n = m, then

$$y_m = m(m-1)(m-2)...... 3.2.1. (ax+b)^0 . a^m = \underline{|m} \, a^m$$

6.13.3 The nth differential co-efficient of $y = \dfrac{1}{ax+b} \left[x \neq -\dfrac{b}{a} \right]$

is given by

$$y_n = \frac{(-1)^n \underline{|n} \, a^n}{(ax+b)^{n+1}}$$

6.13.4 The nth differential co-efficient of y = log (ax+b)

is given by

$$y_n = \frac{(-1)^{n-1} \underline{|n-1} \, a^n}{(ax+b)^n}$$

Cor. If a = 1; b = 0, y = log x, then

$$y_n = \frac{(-1)^{n-1} \underline{|n-1}}{x^n}$$

6.13.5 The nth differential co-efficient of a^{mx}

is $y_n = m^n . a^{mx} (\log a)^n$

Cor. Put a = e, then $y = e^{mx}$ and $y_n = m^n e^{mx}$

6.13.6 The nth differential co-efficient of sin (ax+b)

Let $y = \sin(ax+b)$

then $y_1 = a\cos(ax+b)$

$$= a\sin\left(ax+b+\frac{\pi}{2}\right) \qquad \Big|\ \because \sin\left(\frac{\pi}{2}+\theta\right) = \cos\theta$$

[Note this step]

$$y_2 = a^2\cos\left(ax+b+\frac{\pi}{2}\right) = a^2\sin\left(ax+b+\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^2\sin\left(ax+b+2.\frac{\pi}{2}\right)$$

$$y_3 = a^3\cos\left(ax+b+2.\frac{\pi}{2}\right)$$

$$= a^3\sin\left(ax+b+2.\frac{\pi}{2}+\frac{\pi}{2}\right)$$

$$= a^3\sin\left(ax+b+3.\frac{\pi}{2}\right)$$

....

....

$$y_n = a^n\sin\left(ax+b+n.\frac{\pi}{2}\right)$$

Cor. If $b = 0$, $y = \sin ax$, $y_n = a^n \sin\left(ax+n\frac{\pi}{2}\right)$

6.13.7 Similarly the nth differential co-efficient of $y = \cos(ax+b)$

is $\quad \boldsymbol{y_n = a^n\cos\left(ax+b+n.\frac{\pi}{2}\right)}$

Cor. If $y = \cos ax$

$$y_n = a^n\cos\left(ax+n.\frac{\pi}{2}\right)$$

6.13.8. The nth differential co-efficient of $e^{ax}\sin(bx+c)$

Let $\quad y = e^{ax}\sin(bx+c)$

then $$y_1 = e^{ax} \frac{d}{dx}[\sin(bx+c)] + \sin(bx+c)\frac{d}{dx}[e^{ax}]$$

$$= e^{ax} . \cos(bx+c) . b + \sin(bx+c) . a . e^{ax}$$

$$= e^{ax}[a \sin(bx+c) + b \cos(bx+c)].$$

We determine two constants r and θ, to change the expression into a single sine which will enable us to make the required generlisation by putting, $a = r \cos\theta$, $b = r \sin\theta$

∴ $$r = \sqrt{a^2+b^2}, \qquad \theta = \tan^{-1}\frac{b}{a}$$

Hence $$y_1 = e^{ax}[r \cos\theta . \sin(bx+c) + r \sin\theta . \cos(bx+c)]$$

$$= r\, e^{ax} \sin(bx+c+\theta)$$

Thus y_1 is obtained from y on multiplying it by the constant r and increasing the angle by the constant θ. Repeating the same rule to y_1, we have

$$y_2 = r^2 e^{ax} \sin(bx+c+2\theta)$$

Similarly, $y_3 = r^3 e^{ax} \sin(bx+c+3\theta)$

Hence, in general

$$y_n = \frac{d^n}{dx^n}[e^{ax} \sin(bx+c)] = r^n . e^{ax} \sin(bx+c+n\theta)$$

Puttting values of r and θ,

$$y_n = (a^2+b^2)^{n/2} e^{ax} . \sin\left(bx+c+n . \tan^{-1}\frac{b}{a}\right)$$

6.13.9 Similarly the nth differential co-efficient of $e^{ax}\cos(bx+c)$

is $$y_n = (a^2+b^2)^{n/2} e^{ax} . \cos\left(bx+c+n . \tan^{-1}\frac{b}{a}\right)$$

Example 19. Find the nth differential co-efficient of $\log(ax+x^2)$.

Sol. Let $y = \log(ax+x^2) = \log x(a+x)$

$= \log x + \log(x+a)$

Differentiating n times

$$y_n = \frac{d^n}{dx^n}\log x + \frac{d^n}{dx^n}\log(a+x)$$

$$= \frac{(-1)^{n-1}\,\underline{|n-1}\,.\,1^n}{x^n} + \frac{(-1)^{n-1}\,\underline{|n-1}\,.\,1^n}{(x+a)^n}$$

$$= (-1)^{n-1}\underline{|(n-1)}\left[\frac{1}{x^n} + \frac{1}{(x+a)^n}\right].$$

Example 20. If $y = \cos^3 x$, find y_n.

Sol. We know, $\cos 3x = 4\cos^3 x - 3\cos x$

$\therefore \quad \cos^3 x = \frac{1}{4}[\cos 3x + 3\cos x]$

i.e., $\quad y = \frac{1}{4}[\cos 3x + 3\cos x]$

Differentiating n times

$$y_n = \frac{1}{4}\left[\frac{d^n}{dx^n}\cos 3x + 3\frac{d^n}{dx^n}\cos x\right]$$

$$= \frac{1}{4}\left[3^n\cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right]$$

$$= \frac{1}{4}\left[3^n\cos\left(3x + \frac{n\pi}{2}\right) + 3\cos\left(x + \frac{n\pi}{2}\right)\right]$$

Example 21. Find nth derivative of $\sin^2 x \cos^3 x$.

Sol. Let $y = \sin^2 x \cos^3 x = \sin^2 x \cos^2 x \cdot \cos x$

$$= (\sin x \cos x)^2 \cos x = \frac{1}{4}(2 \sin x \cos x)^2 \cos x$$

$$= \frac{1}{4}(\sin 2x)^2 \cos x = \frac{1}{4}\sin^2 2x \cos x$$

$$= \frac{1}{4}\frac{1-\cos 4x}{2}\cos x \qquad \left| \quad \therefore \sin^2\theta = \frac{1-\cos 2\theta}{2}\right.$$

$$= \frac{1}{8}(\cos x - \cos 4x \cos x) = \frac{1}{16}(2\cos x - 2\cos 4x \cos x)$$

or $$y = \frac{1}{16}[2\cos x - (\cos 5x + \cos 3x)]$$

$[\because 2\cos A \cos B = \cos(A-B) + \cos(A+B)]$

or $$y = \frac{1}{16}[2\cos x - \cos 5x - \cos 3x]$$

$$\therefore \quad y_n = \frac{1}{16}\left[2 \cdot 1^n \cos\left(x + \frac{n\pi}{2}\right) - 5^n \cos\left(5x + \frac{n\pi}{2}\right) - 3^n \cos\left(3x + \frac{n\pi}{2}\right)\right]$$

Example 22. Find nth derivatative of $\frac{1}{x^2-a^2}$

Sol. Let $y = \frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a}$

Multiplying both sides by L.C.M. = (x-a) (x+a),

$$1 = A(x+a) + B(x-a)$$

Put $x-a = 0$, i.e., $x = a$,

$$\therefore \quad 1 = A(a+a) + B(a-a)$$

or $\quad 1 = 2a\,A \quad \therefore \quad A = \dfrac{1}{2a}$

Put $\quad x+a = 0, \quad$ i.e., $x = -a$

$\therefore \quad 1 = A(-a+a)+B(-a-a),$

or $\quad 1 = -2a\,B \quad \therefore \quad B = \dfrac{-1}{2a}$

Putting values of A and B in (1),

$$y = \frac{\frac{1}{2a}}{x-a} - \frac{\frac{1}{2a}}{x+a} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right]$$

$$\therefore \quad y_n = \frac{1}{2a}\left[\frac{d^n}{dx^n}\frac{1}{x-a} - \frac{d^n}{dx^n}\frac{1}{x+a}\right]$$

$$= \frac{1}{2a}\left[\frac{(-1)^n \lfloor n \,.\, 1^n}{(x-a)^{n+1}} - \frac{(-1)^n \lfloor n \,.\, 1^n}{(x+a)^{n+1}}\right]$$

Using $\dfrac{d^n}{dx^n}\left(\dfrac{1}{ax+b}\right) = \left(\dfrac{(-1)^n \lfloor n\, a^n}{(ax+b)^{n+1}}\right)$

$$= \frac{1}{2a}(-1)^n \lfloor n \left[\frac{1}{(x-a)^{n+1}} - \frac{1}{(x+a)^{n+1}}\right]$$

Example 23. Prove that the values of the nth differential co-efficient of $\dfrac{x^3}{(x^2-1)}$ for $x = 0$ is zero, if n is even; and is $-(\lfloor n)$ if n is odd and greater than 1.

Sol. Let $\quad y = \dfrac{x^3}{x^2-1} = \dfrac{x^3}{(x-1)(x+1)}$

or $\quad y = \dfrac{x^3}{(x-1)(x+1)} = x + \dfrac{A}{x-1} + \dfrac{B}{x+1} \quad$ ...(1)

Multiplying by L.C.M. = (x-1) (x+1)

$$x^3 = x(x-1)(x+1) + A(x+1)+B(x-1)$$

Put x = 1, $\quad 1 = A(2) \quad$ or $A = \frac{1}{2}$

Put x = -1, $\quad -1 = B(-2) \quad$ or $B = \frac{1}{2}$

Putting values of A and B in (1),

$$y = x + \frac{1}{2}\left[\frac{1}{x+1} + \frac{1}{x-1}\right].$$

If n>1, the second and higher derivatives of x are 0.

$$\therefore \quad y_n = \frac{1}{2}\left[\frac{(-1)^n \lfloor n}{(x-1)^{n+1}} + \frac{(-1)^n \lfloor n}{(x+1)^{n+1}}\right] \quad ...(1)$$

Case I. When n is even, putting x = 0 in (1),

$$(y_n)_0 = \frac{(+1)\lfloor n}{2}\left[\frac{1}{-1} + \frac{1}{(+1)}\right] = +\frac{\lfloor n}{2}[1-1] = 0$$

Case II. When n is odd

$$(y_n)_0 = \frac{(-1)\lfloor n}{2}\left[\frac{1}{1} + \frac{1}{(+1)}\right] = -\frac{\lfloor n}{2}[1-1] = -\lfloor n$$

6.14 LEIBNITZ'S THEOREM :

This theorem helps us to find the nth differential co-efficient of the product of two functions in terms of the successive derivatives of the functions.

Statements. If u, v be two functions of x, having derivatives of the nth order, then

$$\frac{d^n}{dx^n}(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2\, u_{n-2}\, v_2 + ... + {}^nC_r\, u_{n-r}\, v_r + ... + {}^nC_n\, uv_n\ ,$$

Where suffixes of u and v denote differentiations w.r.t. x.

Proof. We shall prove the theorem by Mathematical Induction.

Set I. Let y = uv

By actual differentiation, we have

$y_1 = u_1 v + u v_1$

and $y_2 = u_2 v + u_1 v_1 + u_1 v_1 + u v_2$

$= u_2 v + 2u_1 v_1 + u v_2$

$= u_2 v + {}^2C_1 u_1 v_1 + {}^2C_2 u v_2$

Thus, the theorem is true for n = 1, 2.

Step II. Let us assume that the theorem is true for a particular value of n, say m, so that, we have

$$y_m = u_m v + {}^mC_1 u_{m-1} v_1 + {}^mC_2\, u_{m-2}\, v_2 + \dots\dots\dots + {}^mC_{r-1}\, u_{m-r+1}\, v_{r-1} + {}^mC_r u_{m-r}\, v_r + \dots\dots\dots + {}^mC_m\, uv_m.$$

Step III. Differentiating both sides, we have

$$y_{m+1} = u_{m+1}\, v + u_m v_1 + {}^mC_1 u_m v_1 + {}^mC_1 u_{m-1} v_2 + {}^mC_2\, u_{m-1}\, v_2 + {}^mC_2\, u_{m-2}\, v_3 + \dots\dots$$

$$+ {}^mC_{r-1} u_{m-r+2} v_{r-1} + {}^mC_{r-1} u_{m-r+1} v_r + {}^mC_r u_{m-r+1} v_r + {}^mC_r u_{m-r} v_{r+1} + \dots + {}^mC_m u_1 v_m + {}^mC_m uv_{m+1.}$$

$$= u_{m+1}\, v + ({}^mC_1 + 1) v_m\, v_1 + ({}^mC_2 + {}^mC_1) u_{m-1}\, v_2 + \dots\dots + ({}^mC_r + {}^mC_{r-1})\, u_{m-r+1}\, v_r + \dots\dots +$$

$${}^mC_m u\, v_{m+1}.$$

But, we know that

$$^mC_{r-1} + {}^mC_r = {}^{m+1}C_r$$

Putting r = 1, 2, 3, ...

$$^mC_o + {}^mC_1 = {}^{m+1}C_r \quad \text{or} \quad 1 + {}^mC_1 = {}^{m+1}C_1$$

$$^mC_1 + {}^mC_2 = {}^{m+1}C_2$$

....

and $\quad ^mC_m = 1 = {}^{m+1}C_{m+1}$

‹ $\quad y_{m+1} = u_{m+1}v + {}^{m+1}C_1 u_m v_1 + {}^{m+1}C_2 u_{m-1} v_2 + ... + {}^{m+1}C_r u_{m-r+1} v_r + ... + {}^{m+1}C_{m+1} u\, v_{m+1}$,

* Changing r to r-1 in $^mC_r\, u_{m-r}\, v_r$ it becomes $^mC_{r-1}\, u_{m-r+1} v_{r-1}$.

which is of exactly the same form as the given formula with

$$n = m+1$$

‹ It is clear that if the theorem is true for n = m, then it is also true for the next higher value n = m+1.

Hence by Mathematical Inductions the theorem is true for all positive integer n.

Example 24. Find the nth derivative of $x^2 \sin x$.

Sol. Let $u = \sin x$ and $v = x^2$

‹ $\quad u_n = \sin\left(x + n\dfrac{\pi}{2}\right)$	$v_1 = 2x$
$u_{n-1} = \sin\left[x + (n-1)\dfrac{\pi}{2}\right]$	$v_2 = 2$
$u_{n-2} = \sin\left[x + (n-2)\dfrac{\pi}{2}\right]$	$v_3 = 0$

Now by Leibnitz's Theorem, we have

$$\frac{d^n}{dx^n}(uv) = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2$$

or $$\frac{d^n}{dx^n}(x^2 \sin x) = \sin\left(x + n\frac{\pi}{2}\right) . x^2 + {}^nC_1 \sin\left[x + (n-1)\frac{\pi}{2}\right] 2x$$

$$+ {}^nC_2 \sin\left[x + (n-2)\frac{\pi}{2}\right] 2$$

$$= x^2 \sin\left(x + \frac{n\pi}{2}\right) + 2nx \sin\left[x + (n-1)\frac{\pi}{2}\right] + n(n-1) \sin\left[x + (n-2)\frac{\pi}{2}\right]$$

Example 25. Find the nth derivative of $e^x \log x$

Let $u = e^x$ and $v = \log x$

∴ $u_n = e^x$ $\qquad v_1 = \frac{1}{x}$

$u_{n-1} = u_{n-2} = ...$ $\qquad v_2 = \frac{-1}{x^2}$

$= u_2 = u_1 = e^x$ $\qquad v_3 = \frac{2}{x^3}$

....

$$v_n = \frac{(-1)^{n-1} \lfloor n-1}{x^n}$$

* ${}^nC_o = 1$, ${}^nC_1 = n$, ${}^nC_2 = \frac{n(n-1)}{\lfloor 2}$, ${}^nC_3 = \frac{n(n-1)(n-2)}{\lfloor 3}$, ${}^nC_n = 1$,

Now by Leibnitz's Theorem, we have

$$\frac{d^n}{dx^n}(e^x \log x) = e^x \log x + {}^nC_1 e^x\left(\frac{1}{x}\right) + {}^nC_2 e^x\left(\frac{-1}{x^2}\right) + ... + {}^nC_n e^x \frac{(-1)^{n-1}\lfloor n-1}{x^n}$$

$$= e^x\left[\log x + \frac{n}{x} - \frac{n(n-1)}{2}\frac{1}{x^2} + ... + \frac{(-1)^{n-1}\lfloor n-1}{x^n}\right]$$

Example 26. If $y = x^2e^x$, show that

$$\frac{d^ny}{dx^n} = \frac{1}{2} n(n-1) \frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{(n-1)(n-2)}{2} y.$$

Sol. $y = x^2e^x = e^x . x^2$

Differentiating n times by Leibnitz's Theorem, we get

$$y_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2$$

or $$y_n = e^x . x^2 + {}^nC_1 e^x . 2x + {}^nC_2 e^x . 2$$

$$= e^x [x^2+2nx+n(n-1)] \quad ...(1)$$

Put n = 1, 2 in (1), successively

$$y_1 = e^x [x^2+2x] \quad ...(2)$$

$$y_2 = e^x [x^2+4x+2] \quad ...(3)$$

Now R.H.S = $\frac{1}{2} n(n-1) \frac{d^2y}{dx^2} - n(n-2)\frac{dy}{dx} + \frac{1}{2}(n-1)(n-2) y$

$$= \frac{1}{2} n(n-1) . e^x [x^2+4x+2] - n(n-2) . e^x (x^2+2x) + \frac{1}{2}(n-1)(n-2) x^2e^x$$

$$= e^x \left[x^2\left\{ \frac{n(n-1)}{2} - n(n-2) + \frac{(n-1)(n-2)}{2} \right\} + x\{2n(n-1) - 2n(n-2)\} + n(n-1)\right]$$

$$= e^x [x^2+2nx+n(n-1)]$$

$= y_n$ from (1).

Example 27. If $y = \sin(m \sin^{-1} x)$, then prove that

$$(1-x^2) y_2 - xy_1 + m^2 y = 0$$

and $$(1-x^2) y_{n+2} = (2n+1) xy_{n+1} + (n^2-m^2)y_n$$

Sol. $y = \sin(m \sin^{-1} x)$

Therefore,

$$y_1 = \cos(m \sin^{-1} x) . \frac{m}{\sqrt{1-x^2}}$$

Cross-multiplying, $\sqrt{1-x^2} \; y_1 = m \cos(m \sin^{-1} x)$

Squaring both sides

$$(1-x^2) y_1^2 = m^2 \cos^2 (m \sin^{-1} x).$$

$$= m^2 [1-\sin^2 (m \sin^{-1} x)]$$

$$| \because \cos^2 \theta = 1 - \sin^2 \theta$$

$$= m^2 (1-y^2) \qquad \text{[By (1)]}$$

Again differentiating both sides w.r.t. x,

$$(1-x^2) \frac{d}{dx} y_1^2 + y_1^2 \frac{d}{dx} (1-x^2) = m^2 \left[0 - \frac{d}{dx} y^2\right]$$

or $$(1-x^2) 2y_1y_2 - 2xy_1^2 = - 2m^2yy_1$$

Dividing every term by $2y_1$ & rearranging or $(1-x^2)y_2 - xy_1 + m^2y = 0$

Now differentiating every term n times by Leibnitz's Theorem, we have

$$(y_2)_n (1-x^2) + {}^nC_1 (y_2)_{n-1} (-2x) + {}^nC_2(y_2)_{n-2} (-2) - [(y_1)_n x + {}^nC_1(y_1)_{n-1} .1] + m^2y_n = 0$$

$$y_{n+2}(1-x^2) + {}^nC_1y_{n+1} (-2x)+{}^nC_2y_n(-2) - y_{n+1} . x - {}^nC_1y_n+m^2y_n = 0$$

$$(1-x^2)y_{n+2} - 2nxy_{n+1} - n(n-1) y_n - xy_{n+1}-ny_n+m^2y_n = 0$$

or $$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$$

Hence $$(1-x^2)y_{n+2} = (2n+1) xy_{n+1}+(n^2-m^2)y_n \qquad ...(II)$$

Note : Determination of the value of the nth derivative of a function for x = 0

The working rule to find $(y_n)_{x=0}$ is being given below:

1. Put the given function equal to y.

2. Find $y_1 = \dfrac{dy}{dx}$

   Then (i) Take L.C.M. (if possible)

   (ii) Square both sides if square Roots are there.

   (iii) Try to get y in R.H.S. (if possible)

3. Again differntiate both sides w.r.t. x to get an equation in y_2, y_1, y.

4. Differentiate both sides n times w.r.t. x by Leibnitz theorem.

5. Put x = 0 in equations of steps 1, 2, 3, 4.

6. Put n = 1, 2, 3, 4 in last equation of step 5.

7. Discuss the two cases when n is even & when n is odd.

Example 28 If $y = (\sin^{-1} x)^2$, prove that :

(i) $(1-x^2)\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 2 = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

Also find the value of nth derivative of y for x = 0.

Sol. $y = (\sin^{-1} x)^2$

Differentiating both sides w.r.t. x,

$$y_1 = 2(\sin^{-1} x) \cdot \frac{1}{\sqrt{1-x^2}} \qquad ...(1)$$

Squaring both sides, $(1-x^2)y_1^2 = 4(\sin^{-1} x)^2 = 4y$

Differentiating again, $2(1-x^2)y_1y_2 - 2xy_1^2 = 4y_1$

Dividing both sides by $2y_1$, $(1-x^2)y_2 - xy_1 - 2 = 0$ ...(2)

Differentiating n times by Leibnitz's Theorem

$(1-x^2)y_{n+2} + {}^nC_1\, y_{n+1}(-2x) + {}^nC_2 y_n(-2) - xy_{n+1} - {}^nC_1 y_n = 0$

or $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$ ...(3)

Put x = 0, in (1), (2) and (3), then

$y_1(0) = 0$ and $y_2(0) = 2$,

and $y_{n+2}(0) = n^2y_n(0)$ ...(4)

Putting n = 1, 2, 3, 4,...... in (4), we get

$y_3(0) = 1^2 \cdot y_1(0) = 0$

$y_4(0) = 2^2 \cdot y_2(0) = 2 \cdot 2^2$

$y_5(0) = 3^2 \cdot y_3(0) = 0$

$y_6(0) = 4^2 \cdot y_4(0) = 2 \cdot 2^2 \cdot 4^2$

$y_7(0) = 5^2 \cdot y_5(0) = 0$

$y_8(0) = 6^2 \cdot y_6(0) = 2 \cdot 2^2 \cdot 4^2 \cdot 6^2.$

....

....

In general,

When n is odd, $y_n(0) = 0$

When n is even, $y_n(0) = 2 \cdot 2^2 \cdot 4^2 \cdot 6^2 ....(n-2)^2.$

Example 29. If $y = \tan^{-1} x$, prove that

$$(1+x^2)y_{n+1}+2nxy_n+n(n-1)y_{n-1} = 0.$$

Hence determine the values of all the derivatives of y with respect to x when x = 0.

Sol. Here $y = \tan^{-1} x$ ...(1)

‹ $y_1 = \dfrac{1}{1+x^2}$ ...(2)

or $y_1(1+x^2) = 1.$

Differentiating n times by Leibnitz's Theorem, we have

$$y_{n+1}(1+x^2)+ny_n \cdot 2x + \frac{n(n-1)}{2} y_{n-1} \cdot 2 = 0$$

or $(1+x^2)y_{n+1}+2nxy_n+n(n-1)y_{n-1} = 0$ ...(3)

Putting x = 0 in (1), (2) and (3), we have

$$y(0) = 0, \qquad y_1(0) = 1$$

and $y_{n+1}(0) = -n(n-1)y_{n-1}(0)$ ...(4)

Putting n = 1, 2, 3, 4.......in 4, we get

$y_2(0) = -1 \cdot (0) \cdot y(0) = 0$

$y_3(0) = -2 \cdot (1) \cdot y_1(0) = -2 = (-1)^1 \lfloor 2$

$y_4(0) = -3 \cdot (2) \cdot y_2(0) = 0$

$y_5(0) = -4 \cdot (3) \cdot y_3(0) = -4 \cdot (3) \cdot (-2) = (-1)^2 \lfloor 4$

... ...

... ...

In general,

When n is even, $y_n(0) = 0$.

When n is odd,

$$y_n(0) = (-1)^{\frac{n-1}{2}} \lfloor n-1.$$

6.15 SELF ASSESSMENT QUESTIONS :

1. Write down the derivative of the following functions :

(i) X^9 (ii) x^{-8} (iii) $x^{9/2}$

(iv) $(2x+5)^3$ (v) $\frac{1}{\sqrt{3+x}}$

2. Differenitiate from first principle

(i) $\frac{1}{\sqrt{x}}$ (ii) $\sqrt{ax+b}$

3. Find $\frac{dy}{dx}$ for the following

(i) $y = \sqrt{x} - \frac{1}{\sqrt{x}}$

(ii) $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$

(iii) $y = \left(x - \frac{1}{x}\right)^2 \left(x + \frac{1}{x^2}\right)$

(iv) $y = \frac{1-\sqrt{x}}{1+\sqrt{x}}$

4. Find $\frac{dy}{dx}$ if $y = \mathbf{v}^3 + 2\mathbf{v}^2 + 5$, $\mathbf{v} = 3u+1$ & $u = 9x+1$

5. Differentiati wrt x (i) $(x + \sqrt{x^2+2})^n$

(ii) $\sqrt{\dfrac{a^2+x^2}{a^2-x^2}}$

6. Differentiate the following wrt x

(i) $(1+x^2)\,\text{Cos}\,x$ (ii) $x\,\text{Cosec}\,x$

(iii) $(x+\sin x)(x+\cos x)$ (iv) $\sqrt{\dfrac{1+\text{Cos}\,x}{1-\text{Cos}\,x}}$

7. Find $\dfrac{dy}{dx}$, if

(i) $y = \log\left(\sqrt{x+a} + \sqrt{x-b}\right)$

(ii) $y = \log\sqrt{\dfrac{1-\text{Cos}x}{1+\text{Cos}x}}$

8. Find $\dfrac{dy}{dx}$ if $(x^2+y^2)^2 = xy$

9. Find $\dfrac{dy}{dx}$ if $x^2+y^2 = \log xy$

10. Find $\dfrac{dy}{dx}$ if $x = a\theta^2,\ y = 2a\theta$

11. Find $\dfrac{dy}{dx}$ if $x = a(\theta+\text{Sin}\theta),\ y = a(1+\text{Cos}\theta)$

12. Find $\dfrac{d^2y}{dx^2}$ at $\theta = \pi/2$ if $x = a(\theta+\sin\theta),\ y = a(1+\cos\theta)$.

13. If $y = \left\{x+\sqrt{x^2+1}\right\}^m$, show that

$\{x^2+1)\,y_2 + xy_1 - m^2y = 0$.

6.16 KEY WORDS :

Limit of a Function, Differential coefficient, Implicit differentiation, parametric differentiation, Successive differentiation.

6.17 SUGGESTED READINGS :

1. Grewal, B.S. - Engineering Mathematics

2. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.

3. Ramana, B.V. - Higher Engineering Mathematics.

4. Aggarwal, R.S. - Modern Approach of Mathematics.

E E E

Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**	
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No.**	**:**	**07**
Lesson	**:**	**Integral Calculus**			

7.0 OBJECTIVES :

In this lesson, you will be able to understand

* to find the indefinite integral of a given function.

* to state the standared indefnite integral using substitution, parts, partial fractions.

* to evaluate definite integral.

7.1 INTRODUCTION :

Previously, we dealt with the methods of findings derivatives of a given function. We had noticed that the derivatives so obtained were also the functions. Here, we propose to deal with the converse. Consider the following examples :

(i) If $f(x) = x$, then $f'(x) = 1$

(ii) If $f(x) = x^{-3}$, then $f'(x) = -3x^{-4}$

(iii) If $f(x) = x^{5/2}$, then $f'(x) = \frac{5}{2}x^{3/2}$

(iv) If $f(x) = \sin x$, then $f'(x) = \cos x$

Now let us consider the questions :

(i) What is the function whose derivative is 1 ?

(ii) What is the function whose derivative is $-3x^{-4}$?

(iii) What is the function whose derivative is $\frac{5}{2}x^{3/2}$?

(iv) What is the function whose derivative is $\cos x$?

Clearly the answers to these questions are x, x^{-3}, $x^{5/2}$ and $\sin x$ respectively. The functions which we find are called primitives or anti-derivatives or integrals of the given function. Thus we are given a function of x and we try to find another function whose derivative is always the given function. This is exactly the problem of integral calculus. "Integration", therefore, is called the inverse process of differentiation. For example, the function whose derivative is $\cos x$ is $\sin x$.

‹ $\sin x$ is called the primitive or anti-derivative or the integral of $\cos x$.

Definition. If g(x) be any differentiable function of x such that

$$\frac{d}{dx}[g(x)] = f(x)$$

then g(x), is called an anti-derivative or a primitive or an indefinite integral or simply an integral of f(x).

Symbolically,

$$g(x) = \int f(x)dx$$

and is read as "g(x) is the integral of f(x) w.r.t. x".

This process of finding the integral of a given function is called integration and the given function is called the integrand.

Remarks :

1. The symbol $\int$ dx is purely a symbol of operation, which means integral of with respect to x, $\int$ and dx mean nothing when taken separately.

2. From the definition of anti-derivative it is clear that

$$\frac{d}{dx}\left[\int f(x)\, dx\right] = f(x)$$

7.2 INDEFINITE INTEGRAL :

Suppose $f(x) = x^2$, $g(x) = x^2+9$ and $p(x) = x^2+c$, where c is a constant. By differentiating these functions, we get

$$\frac{d}{dx}f(x) = \frac{d}{dx}g(x) = \frac{d}{dx}p(x) = 2x$$

$$\therefore \qquad \int 2x\, dx = x^2 \text{ or } x^2+9 \text{ or } x^2+c$$

Hence $\int 2xdx$ does not give a definite value and is called an indefinite integral. The general value of $\int 2x\, dx = x^2+c$, where c is an arbitrary constant. The constant added with the integral is called the constant of integration.

For example

$$(\sin x) = \cos x, \ \therefore \int \cos x \, dx = \sin x + c$$

From these considerations, we conclude that integral of a function is not unique and that if f(x) be any one integral of g(x), then (i) f(x)+c is also its integral, c being any constant. (ii) every integral of g(x) can be obtained from f(x)+c, by giving a suitable value to c.

7.3 RULES OF INTEGRATION :

Rule I. *The integral of the product of a constant and a function is equal to the product of the constant and integral of the function, i.e.,*

$$\int k\, f(x)\, dx = k \int f(x)\, dx,$$

where k is some constant.

Proof $\quad \frac{d}{dx}\left[k \int f(x)\, dx\right] = k \frac{d}{dx} \int f(x)\, dx = k\, f(x)$

$\therefore$ From the definition, we have

$$k \int f(x)\, dx = \int k\, f(x)\, dx$$

Rule II. *The integral of the sum or difference of functions is equal to the sum or difference of their integrals. Symbolically*

$$\int [f_1(x)+f_2(x)+..........+f_n(x)]\, dx$$

$$= \int f_1(x)\, dx + \int f_2(x)\, dx + ... + \int f_n(x)\, dx$$

where $f_1(x), f_2(x), ..., f_n(x)$ are functions of x.

Proof. $\quad \frac{d}{dx}\left[\int f_1(x)\, dx + \int f_2(x)\, dx + ... + \int f_n(x)\, dx\right]$

$$= \frac{d}{dx}\int f_1(x)\,dx + \frac{d}{dx}\int f_2(x)\,dx + ... + \frac{d}{dx}\int f_n(x)\,dx$$

$$= f_1(x)+f_2(x)+...+f_n(x)$$

$\therefore$ From the definition, we have

$$\int\left[f_1(x) + f_2(x)+...+f_n(x)\right] dx$$

$$= \int f_1(x)\,dx+ \int f_2(x)\,dx+...+\int f_n(x)\,dx$$

7.4 SOME STANDARD RESULTS :

We give some standard results using derivatives of some well-known functions :

I. $\because \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n,$ $\quad\therefore \int x^n\,dx = \frac{x^{n+1}}{n+1} + c,\ n \neq -1$

II. $\because \frac{d}{dx}(\log x) = \frac{1}{x}$ $\quad\therefore \int \frac{1}{x}\,dx = \log x + c$

III. $\because \frac{d}{dx}(\sin x) = \cos x,$ $\quad\therefore \int \cos x\,dx = \sin x + c$

IV. $\because \frac{d}{dx}(\cos x) = -\sin x,$ $\quad\therefore \int \sin x\,dx = -\cos x + c$

V $\because \frac{d}{dx}(\tan x) = \sec^2 x,$ $\quad\therefore \int (\sec^2 x)\,dx = \tan x + c$

VI $\because \frac{d}{dx}(\cot x) = -\text{cosec}^2 x,$ $\quad\therefore \int (\text{cosec}^2 x)\,dx = -\cot x + c$

VII $\because \frac{d}{dx}(\sec x) = \sec x \tan x,$ $\quad\therefore \int \sec x \tan x\,dx = \sec x + c$

VIII $\because \frac{d}{dx}(\text{cosec } x) = -\text{cosec } x \cot x,$

$\therefore \int (\text{cosec } x \cot x\,dx = -\text{cosec } x + c$

IX $\because \frac{d}{dx}(e^x) = e^x,$ $\quad\therefore \int (e^x)\,dx = e^x + c$

X. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \Rightarrow \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x+c = -\cos^{-1}x+c$

XI. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \Rightarrow \int \frac{dx}{1+x^2} = \tan^{-1} x+c = -\cot^{-1}x+c$

XII. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}} \Rightarrow \int \frac{dx}{x\sqrt{x^2-1}} = (\sec^{-1} x)x+c = -\cos^{-1}x+c$

XIII. $\frac{d}{dx}\left(\frac{a^x}{\log a}\right) = a^x, \Rightarrow \int a^x\, dx = \frac{a^x}{\log a} +c$

XIV $\frac{d}{dx}\left(\frac{(ax+b)^{n+1}}{a(n+1)}\right) = (ax+b)^n \Rightarrow \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} +c, n \neq -1$

XV $\frac{d}{dx}\left(\frac{\log (ax+b)}{a}\right) = \frac{1}{ax+b} \Rightarrow \int \frac{dx}{ax+b} = \frac{1}{a}\log|ax+b|+c$

Example 1.

(i). $\int 5x^2\, dx = 5\int x^2\, dx = 5\,\frac{x^{2+1}}{2+1} = \frac{5}{3}\, x^3+c$

(ii). $\int (3-2x-x^4)\, dx = 3\int dx - 2\int x\, dx - \int x^4\, dx$

$= 3x - 2\,\frac{x^{1+1}}{1+1} - \frac{x^{4+1}}{4+1}$

$= 3x - x^2 - \frac{x^5}{5} +c$

(iii). $\int (x^2-1)^2\, dx = \int (x^4-2x^2+1)\, dx$

$\int x^4\, dx - 2\int x^2\, dx + \int dx$

$= \frac{x^5}{5} - \frac{2}{3}x^3 + x+c$

(iv). $\int\left(\sqrt{x} - \frac{1}{2}x + \frac{2}{\sqrt{x}}\right)dx$

$= \int\sqrt{x}\, dx - \frac{1}{2}\int x\, dx + 2\int\frac{1}{\sqrt{x}}\, dx$

$$= \frac{2}{3}x^{3/2} - \frac{1}{4}x^2 + 4x^{1/2} + c$$

(v). $\int \frac{x^4+1}{x^2}\,dx$

$$= \int (x^2+x^{-2})\,dx = \int x^2\,dx + \int x^{-2}\,dx$$

$$= \frac{x^3}{3} - \frac{1}{x} + c$$

Example 2. Integrate $\left(x - \frac{1}{x}\right)^3$ w.r.t. x.

Solution. $I = \int \left(x - \frac{1}{x}\right)^3 dx = \int \left(x^3 - 3x + \frac{3}{x} - \frac{1}{x^3}\right) dx$

$$= \int (x^3 - 3x + 3x^{-1} - x^{-3})\,dx$$

$$= \int x^3\,dx - 3\int x\,dx + 3\int x^{-1}\,dx - \int x^{-3}\,dx$$

$$= \frac{1}{4}x^4 - \frac{3}{2}x^2 + 3\log x + \frac{1}{2}x^{-2} + C$$

Example 3. Integrate $\frac{x^2-3x+\sqrt[3]{x}+7}{\sqrt{x}}$ w.r.t. x.

Solution. $I = \int \left(x^{3/2} - 3x^{1/2} + x^{-1/6} + 7x^{-1/2}\right) dx$

$$= \frac{x^{5/2}}{\frac{5}{2}} - 3 \cdot \frac{x^{3/2}}{\frac{3}{2}} + \frac{x^{5/6}}{\frac{5}{6}} + 7 \cdot \frac{x^{1/2}}{\frac{1}{2}}$$

$$= \frac{2}{5}x^{5/2} - 2x^{3/2} + \frac{6}{5}x^{5/6} + 14x^{1/2} + c$$

Example 4. Evaluate $\int \frac{\sin x}{1+\sin x}\, dx$

Solution. $I = \int \frac{\sin x(1-\sin x)}{1-\sin^2 x} = \int \frac{\sin x-\sin^2 x)}{\cos^2 x}\, dx$

$$= \int \frac{\sin x}{\cos^2 x}\, dx - \int \frac{\sin^2 x}{\cos^2 x}\, dx$$

$$= \int \sec x \tan x\, dx - \int \tan^2 x\, dx$$

$$= \sec x - \int (\sec^2 x - 1)\, dx$$

$$= \sec x - \tan x + x + c$$

7.5 INTEGRATION BY SUBSTITUTION :

Integration can easily be obtained by the substitution of a new variable for the given independent variable, in other words, by changing the independent variable. It can be seen with the following illustrations.

Example 5.

Solution : Evaluate $\int \cos^3 x\, dx$

Let $\sin x = t \Rightarrow \cos x \frac{dx}{dt} = 1$

$$\therefore \int \cos^3 x\, dx = \int \cos^3 x \frac{dx}{dt} . dt$$

$$= \int \cos^2 x . \cos x \frac{dx}{dt} . dt$$

$$= \int (1-\sin^2 x) . dt = \int dt - \int t^2 dt = t - \frac{t^3}{3}$$

$$= \sin x - \frac{1}{3}\sin^3 x + c$$

Example 6. Evaluate $\int (4x+5)^6\, dx$

Solution : Let $4x+5 = t \Rightarrow 4\,\dfrac{dx}{dt} = 1$ or $\dfrac{dx}{dt} = \dfrac{1}{4}$

$$\therefore \quad \int (4x+5)^6\, dx = \int (4x+5)^6 \frac{dx}{dt}\,.\,dt$$

$$= \int t^6 \,.\, \frac{1}{4}\, dt = \frac{1}{4}\int t^6\, dt = \frac{1}{4}\,.\,\frac{t^7}{7}$$

$$= \frac{1}{28}\,.\,(4x+5)^7 + c$$

Example 7. Evaluate $\int x\,(x^2+4)^5\, dx$.

Solution :

Put $x^2+4 = t \Rightarrow 2x\,\dfrac{dx}{dt} = 1$ or $x\,\dfrac{dx}{dt} = \dfrac{1}{2}$

$$\therefore \quad \int x(x^2+4)^5\, dx = \int (x^2+4)^5\, x\,\frac{dx}{dt}\,.\,dt$$

$$= \int t^5 \,.\, \frac{1}{2}\, dt = \frac{1}{12}\, t^6$$

$$= \frac{1}{12}\,(x^2+4)^6 + c$$

Example 8. Evaluate $\int \sin x \cos^3 x\, dx$.

Solution :

Put $\cos x = t \Rightarrow -\sin x\,\dfrac{dx}{dt} = 1$

$$\therefore \quad \int \sin x \cos^3 x\, dx = -\int \cos^3 x\,(-\sin x)\,\frac{dx}{dt}\,dt$$

$$= -\int t^3\,.\,1\, dt = -\frac{t^4}{4} = -\frac{\cos^4 x}{4} + c$$

Example 9. Integrate the following functions w.r.t.x,

(a) $(x^3+2)^2.3x^2$, (b) $\dfrac{8x^3}{(x^3+2)^3}$

Solution. Let us put $(x^3+2)=t$, then $dt = 3x^2dx$

(a) $\int (x^3+2)^2 3x^2 dx = \int t^2 dt = \dfrac{t^2}{3} = \dfrac{1}{3} . (x^3+2)^3 + c$

(b) $\int \dfrac{8x^2}{(x^3+2)^3} dx = 8 \dfrac{1}{3} \int (x^3+2)^{-3}\, 3x^2\, dx$

$= \dfrac{8}{3} \int t^{-3}\, dt = \dfrac{8}{3} \left(- \dfrac{1}{2} . t^{-2}\right)$

$= \dfrac{4}{3(x^3+2)^2} + c$

Example 10. Integrate $e^{\tan x} \sec^2 x$ w.r.t., x,

Solution. We put $\tan x = t$

$\sec^2 x \dfrac{dx}{dt} = 1$, i.e.,$\sec^2 x\, dx = dt$

$\therefore \int e^{\tan x} \sec^2 x\, dx = \int e^t\, dt = e^t = e^{\tan x} + c$

Example 11. Evaluate $\int \dfrac{x^5 dx}{1+x^{12}}$

Solution. We put $x^6 = t \Rightarrow 6x^5 \dfrac{dx}{dt} = 1$,i.e. $6x^5dx = dt$

$\therefore \int \dfrac{x^5}{1+x^{12}} dx \quad \int \dfrac{dt}{6(1+t^2)} = \dfrac{1}{6} \tan^{-1} t = \dfrac{1}{6} \tan^{-1} x^6 + c$

Example 12. Integrate $\sqrt{\dfrac{1}{x}} \sin \sqrt{x}$ w.r.t.x

Solution Put $\sqrt{x} = t$, then $\dfrac{1}{2\sqrt{x}} dx = dt$, i.e.,$\dfrac{dx}{\sqrt{x}} = 2dt$

$\therefore \quad I = \int 2 \sin t \, dt = -2 \cos t = -2 \cos \sqrt{x} \, . + c$

Example 13. Integrate $\frac{x^3}{(x^2+1)^3}$ w.r.t. x,

Solution. Put $x^2+1 = t$ so that $2x\,dx = dt$

$$I = \frac{1}{2}\int \frac{x^2.2x\,dx}{(x^2+1)^3} = \frac{1}{2}\int \frac{(t-1)dt}{t^3}$$

$$= \frac{1}{2}\int \left(\frac{1}{t^2} - \frac{1}{t^3}\right) dt = \frac{1}{2}\left(-\frac{1}{t} + \frac{1}{2t^2}\right)$$

$$= \frac{1}{2}\left(\frac{1-2t}{2t^2}\right) = \frac{1}{2}\left[\frac{1-2(x^2+1)}{2(x^2+1)^2}\right]$$

$$= -\frac{1}{4} \cdot \frac{2x^2+1}{(x^2+1)^2} + c$$

7.6 SOME MORE RESULTS :

I $\int \tan x \, dx = \log \sec x$

II $\int \operatorname{cosec} x \, dx = \log \sin x$

III $\int \operatorname{cosec} x \, dx = \log \tan \left(\frac{x}{2}\right) = \log (\operatorname{cosec} x - \cot x)$

IV $\int \sec x \, dx = \log \tan \left(\frac{x}{2} + \frac{\pi}{4}\right) = \log (\sec x + \tan x)$

V $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}$ [Hint :Put $x = a \sin \theta$]

VI $\int \frac{dx}{\sqrt{x^2-a^2}} = \log \left\{\frac{x+\sqrt{x^2-a^2}}{a}\right\}$ [Hint : put $x= a \sec \theta$]

VII $\int \frac{dx}{\sqrt{x^2+a^2}} = \log \left\{\frac{x+\sqrt{x^2+a^2}}{a}\right\}$ [Hint : Put $x=a \tan \theta$]

VIII $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$

IX $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$

X $\int \frac{dx}{x^2+a^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right)$ [Hint Put $x=a \tan \theta$]

7.7 INTEGRATION BY PARTS :

Let u and v be two functions of x. From differential calculus, we have

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrating both sides w.r.t. x, we have

$$uv = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

Transposing, we get $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

If $u = f(x)$ and $\frac{dv}{dx} = g(x)$, so that $v = \int g(x)\,dx$,

the above rule may be written as

$$\int f(x)\,g(x)\,dx = f(x)\left(\int g(x)\,dx\right) - \int \left\{\frac{d}{dx}f(x)\right\}\left\{\int g(x)\,dx\right\}dx$$

Thus

The integral of the product of two functions = first function x integral of the second function - integral of (the derivative of the first x integral of the second function).

Exmaple 14. Integrate $x^2 \sin x$, w.r.t.x

Solution, Let x^2 be the first function and sin x be the second function, Then.

$$I = x^2 \int \sin x\,dx - \int\left(\frac{d}{dx}x^2\right).\left(\int \sin x\,dx\right)dx$$

$$= -x^2 \cos x + \int 2x \cos x\,dx$$

Integrating by parts, the second term of the R.H.S.,we get

$$I = -x^2 \cos x + 2x \int \cos x\,dx - \int \frac{d}{dx}(2x)\left(\int \cos x\,dx\right)dx$$

$$= -x^2 \cos x + 2x \sin x - 2\int \sin x\,dx$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

Example 15. Evaluate $\int x^2 e^{3x} dx$

Solution. Let x^2 be the first function and e^{3x} be the second one.Then

$$I = x^2 \int e^{3x}\,dx - \int \left\{\frac{d}{dx}(x^2).\int e^{3x}\,dx\right\} dx$$

$$= \frac{x^2e^{3x}}{3} - \int 2x.\frac{e^{3x}}{3}\,dx$$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\int x^{e3x}\,dx$$

Integrating by parts the second member on the R.H.S., taking x as the first function, we have

$$I = \frac{x^2e^{3x}}{3} - \frac{2}{3}\left\{x\int e^{3x}\,dx - \int \frac{d}{dx}(x).\int e^{3x}\,dx\right)dx\right\}$$

$$= \frac{x^2e^{3x}}{3} - \frac{2}{3}\left\{x.\frac{e^{3x}}{3} - \frac{1}{3}\int e^{3x}\,dx\right\}$$

$$= \frac{x^2e^{3x}}{3} - \frac{2xe^{3x}}{9} + \frac{2}{27}e^{3x} + c$$

Example16. Evaluate (a) $\int \log x\,dx$, (b) $\int x^n \log x\,dx$

Solution. (a) Take log x as the first function and 1 as the second.

$$\int \log x\,.\,1\,dx = \log x \int 1.\,dx - \int\left[\frac{d}{dx}(\log x)\,.\int 1.\,dx\right]dx$$

$$= x \log x - \int \frac{1}{x}.\,x\,dx$$

$$= x \log x - x,$$

(b) Taking log x as the first function, we have

$$I = \log x \int x^n\,dx - \int \left[\frac{d}{dx}(\log x) \cdot \int x^n\,dx \right] dx$$

$$= \log x \cdot \frac{x^{n+1}}{n+1} - \int \frac{1}{x} \cdot \frac{x^{n+1}}{n+1}\,dx$$

$$= \log x \cdot \frac{x^{n+1}}{n+1} - \frac{x^{n+1}}{(n+1)^2}$$

Example 17. Evaluate $\int x \tan^{-1} x\,dx$.

Solution. Let $\tan^{-1} x$ be the first function, then

$$I = \tan^{-1} x \int x\,dx - \int \left[\frac{d}{dx}(\tan^{-1} x) \cdot \int x\,dx \right] dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{1+x^2} \frac{x^2}{2}\,dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{(x^2+1)-1}{x^2+1}\,dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int dx + \frac{1}{2} \int \frac{1}{x^2+1}\,dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \tan^{-1} x$$

Example 18. Integrate the function $x \cos^2 x$ w.r.t. x.

Solution. From trigonometry, $\cos^2 x = \frac{1+\cos 2x}{2}$

$$\therefore \quad I = \int x \left(\frac{1+\cos 2x}{2} \right) dx = \frac{1}{2} \int x\,dx + \frac{1}{2} \int x \cos 2x\,dx$$

The second integral is integrable by parts.

$$\therefore \quad \int x \cos^2 x\,dx = \frac{x^2}{4} + \frac{1}{2} \left[x \cdot \frac{\sin 2x}{2}\,dx - \int \frac{\sin 2x}{2}\,dx \right]$$

$$= \frac{x^2}{4} + \frac{x \sin 2x}{4} + \frac{1}{8} \cos 2x$$

Example 19. Evaluate the following integrals :

(i) $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx$

(ii) $\displaystyle\int e^x \cdot \frac{1+\sin x}{1+\cos x}\,dx.$

Solution. (i) $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx = \int \frac{(x+1)-1}{(x+1)^2}\,e^x\,dx$

$$= \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] e^x\,dx$$

Integrating $\dfrac{1}{x+1}e^x\,dx$ by parts, we get

$$\int \frac{1}{x+1}\,e^x\,dx = \frac{1}{x+1}\cdot e^x - \int \frac{-1}{(x+1)^2}\,e^x\,dx$$

$$\Rightarrow \quad \int \left[\frac{1}{x+1} - \frac{1}{(x+1)^2}\right] e^x\,dx = \frac{1}{x+1}\cdot e^x$$

(ii) $$e^x\left(\frac{1+\sin x}{1+\cos x}\right) = e^x\left\{\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\right\}$$

$$= e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right\}$$

Integrating $e^x\tan\dfrac{x}{2}$ by parts, we get

$$\int e^x\tan\frac{x}{2}\,dx = e^x\tan\frac{x}{2} - \int e^x\,\frac{1}{2}\sec^2\frac{x}{2}\,dx$$

$$\Rightarrow \quad \int e^x\left(\frac{1}{2}\sec^2\frac{x}{2} + \tan\frac{x}{2}\right)dx = e^x\tan\frac{x}{2}$$

Example 20. Evaluate $\int \frac{x^2 \tan^{-1} x}{1+x^2} dx$ by parts, we get

Solution. Let $\tan^{-1} x = \theta$, i.e., $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$

$$\therefore \quad I = \int \frac{\tan^2\theta \,.\, \theta \sec^2\theta}{1+\tan^2\theta} d\theta = \int \frac{\theta \tan^2\theta \,.\, \sec^2\theta}{\sec^2\theta} d\theta$$

$$= \int \theta \tan^2\theta\, d\theta = \int \theta (\sec^2\theta - 1)\, d\theta$$

$$= \int \theta \sec^2\theta\, d\theta - \int \theta\, d\theta$$

$$= \theta \tan\theta - \int 1 \,.\, \tan\theta\, d\theta - \frac{\theta^2}{2}$$

$$= \theta \tan\theta - \log \sec\theta - \frac{1}{2}\theta^2$$

$$= \theta \tan\theta - \log\sqrt{1+\tan^2\theta} - \frac{1}{2}\theta^2$$

$$= x \tan^{-1} x - \frac{1}{2}\log(1+x^2) - \frac{1}{2}[\tan^{-1} x]^2$$

7.8 INTEGRATION BY PARTIAL FRACTIONS :

It is explained by the following examples

Example 21. Integrate $\frac{x}{(x-1)(2x+1)}$ w.r.t. x.

Solution. Let $\frac{x}{(x-1)(2x+1)} = \frac{A}{(x-1)} + \frac{B}{(2x+1)}$

Multiplying both sides by (x-1)(2x+1), we have

$$x = A(2x+1)+B(x-1)$$

Putting $x = 1$ and $-\frac{1}{2}$, we get

$$1 = 3A \quad \Rightarrow \quad A = \frac{1}{3}$$

and $\quad -\frac{1}{2} = -\frac{3}{2}B \quad \Rightarrow \quad B = \frac{1}{3}$

$$\therefore \quad \int \frac{x}{(x-1)(2x+1)}\,dx = \int \left[\frac{1}{3(x-1)} + \frac{1}{3(2x+1)}\right] dx$$

$$= \frac{1}{3}\int \frac{1}{(x-1)}\,dx + \frac{1}{3}\int \frac{1}{(2x+1)}\,dx$$

$$= \frac{1}{3}\log(x-1) + \frac{1}{3}\cdot\frac{1}{2}\log(2x+1)$$

Example 22. Evaluate $\int \frac{dx}{x-x^3}$.

Solution. $\quad \int \frac{dx}{x-x^3} = \int \frac{dx}{x(1-x^2)} = \int \frac{dx}{x(1-x)(1+x)}$

Let $\quad \frac{1}{x(1-x)(1+x)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x}$

Multiplying both sides by x(1-x)(1+x), we have

$$1 = A(1-x)(1+x)+Bx(1+x)+Cx(1-x)$$

Putting x = 0, 1 and -1, we have

$1 = A \quad \Rightarrow \quad A = 1$

$1 = 2B \quad \Rightarrow \quad B = \frac{1}{2}$

and $\quad 1 = -2C \quad \Rightarrow \quad C = -\frac{1}{2}$

$$\therefore \quad \int \frac{dx}{x-x^3} = \int \left\{\frac{1}{x} + \frac{1}{2(1-x)} - \frac{1}{2(1+x)}\right\} dx$$

$$= \log x - \frac{1}{2}\log(1-x) - \frac{1}{2}\log(1+x)$$

$$= \frac{1}{2}[2\log x - \log(1-x) - \log(1+x)]$$

$$= \frac{1}{2}\log\left\{\frac{x^2}{1-x^2}\right\}$$

Example 23. Evaluate $\int \frac{x^3}{(x-a)(x-b)(x-c)}\, dx$

Solution. Let $\frac{x^3}{(x-a)(x-b)(x-c)} = 1 + \frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$

Multiplying both sides by (x-a)(x-b)(x-c), we have

$$x^3 = (x-a)(x-b)(x-c)+(x-b)(x-c)A+(x-c)(x-a)B+(x-a)(x-b)C$$

Putting x = a, b and c, we get

$$A = \frac{a^3}{(a-b)(a-c)},\quad B = \frac{b^3}{(b-a)(b-c)},\quad C = \frac{c^3}{(c-a)(c-b)}$$

$$\therefore \quad I = \int dx + A\int \frac{dx}{(x-a)} + B\int \frac{dx}{(x-b)} + C\int \frac{dx}{(x-c)}$$

$$= x + A \log(x-a) + B \log(x-b) + C \log(x-c)$$

where A, B, C have the values obtained above.

Example 24. Evaluate $\int \frac{dx}{\sin x + \sin 2x}$

Solution. Let $I = \int \frac{dx}{\sin x + \sin 2x} = \int \frac{dx}{\sin x + 2 \sin x \cos x}$

$$= \int \frac{dx}{\sin x(1+2\cos x)} = \int \frac{\sin x\, dx}{\sin^2 x(1+2\cos x)}$$

$$= \int \frac{\sin x\, dx}{(1-\cos^2 x)(1+2\cos x)}$$

$$= \int \frac{\sin x\, dx}{(1-\cos x)(1+\cos x)(1+2\cos x)}$$

Put cos x = t so that sin x dx = -dt

$$\therefore \quad I = -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$

Let $\frac{1}{(1-t)(1+t)(1+2t)} \equiv \frac{A}{1-t} + \frac{B}{1+t} + \frac{C}{1+2t}$

$$1 = (1+t)(1+2t)A+(1-t)(1+2t)B+(1-t)(1+t)C$$

Putting t=1, -1 and $-\frac{1}{2}$ respectively, we have

$$I = 6A \quad \Rightarrow \quad A = \frac{1}{6}$$

$$I = -2B \quad \Rightarrow \quad B = -\frac{1}{2}$$

and $\quad I = \frac{3}{4}C \quad \Rightarrow \quad C = \frac{4}{3}$

$$\therefore \quad I = -\int\left[\frac{1}{6(1-t)} - \frac{1}{2(1+t)} + \frac{4}{3(1+2t)}\right] dt$$

$$= \frac{1}{6}\log(1-t)+\frac{1}{2}\log(1+t)-\frac{2}{3}\log(1+2t)$$

$$= \frac{1}{6}\log(1-\cos x)+\frac{1}{2}\log(1+\cos x) - \frac{2}{3}\log(1+2\cos x)$$

Example 25. Integrate $\frac{x+5}{(x+1)(x+2)^2}$ w.r.t. x.

Solution. Let $\frac{x+5}{(x+1)(x+2)^2} = \frac{A}{(x+1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$

$$\Rightarrow \quad x+5 \equiv A(x+2)^2+B(x+1)(x+2)+C(x+1)$$

Putting x = -2 and x = -1, we get A = 4, C = -3.

Comparing co-efficient of x^2, we get A+B = 0, i.e., B = -4

$$\therefore \quad I = 4\int\frac{dx}{x+1} - 4\int\frac{dx}{x+2} - 3\int\frac{dx}{(x+2)^2}$$

$$= 4\log(x+1)-4\log(x+2) + \frac{3}{x+2}$$

7.9 DEFINITE INTEGRALS :

In geometrical and other applications of Integral Calculus, it becomes necessary to find the difference in the values of the integral of a function f(x) between two assigned values of an independent variable x, say, a, b. The difference is called the Definite Integral f(x), over the interval [a, b] and is denoted by

$$\int_a^b f(x)\,dx$$

Thus $$\int_a^b f(x)\,dx = g(b)-g(a),$$

where g(x) is an integral of f(x).

The difference [g(b)-g(a)] is sometimes denoted as

$$\Big[\,g(x)\,\Big]_a^b$$

Thus if g(x) is an integral of f(x), we write

$$\int_a^b f(x)\,dx = \Big[\,g(x)\,\Big]_a^b = g(b)-g(a)$$

The numbers a and b are respectively called the lower limit and the upper limit of definite integral. $\int_a^b f(x)\,dx$ is called the definite integral because the indefinite constant of integration does not appear in it. Since

$$f(x)\,dx = \Big[g(x)+c\Big]_a^b = \Big\{g(b)+c\Big\} - \Big\{g(a)+c\Big\}$$
$$= g(b)-g(a)$$

so that arbitrary constant c disappears in the process.

Example 26.

i). $$\int_{-1}^{1}(2x^2-x^3)\,dx = \left[\frac{2x^3}{3} - \frac{x^4}{3}\right]_{-1}^{1} = \left(\frac{2}{3} - \frac{1}{4}\right) - \left(-\frac{2}{3} - \frac{1}{4}\right)$$

$$= \frac{4}{3}.$$

ii). $$\int_2^4 (3x-2)^2\,dx = \int_2^4 (9x^2-12x+4)\,dx$$

$$= \left[9 \cdot \frac{x^3}{3} - 12 \cdot \frac{x^2}{2} + 4x \right]_2^4$$

$$= \left[3x^3 - 6x^2 + 4x \right]_2^4$$

$$= (192-96+16)-(24-24+8) = 104$$

iii). $\int_{-3}^{-1} \left[\frac{1}{x^2} - \frac{1}{x^3} \right] dx = \left[-\frac{1}{x} + \frac{1}{2x^3} \right]_{-3}^{-1} = \left[1 + \frac{1}{2} \right] - \left[\frac{1}{3} + \frac{1}{18} \right]$

$$= \frac{10}{9}$$

iv). $\int_{1}^{4} \frac{dx}{\sqrt{x}} = \left[2\sqrt{x} \right]_1^4$

$$= 2(\sqrt{4} - \sqrt{1}) = 2$$

v). $\int_{6}^{10} \frac{dx}{x+2} = \left[\log(x+2) \right]_6^{10}$

$$= \log 12 - \log 8 = \log \left(\frac{12}{8} \right) = \log \left(\frac{3}{2} \right)$$

vi). $\int_{1}^{e} \log x \, dx = \left[x \log x - x \right]_1^e$

$$= (e \log e - e) - (\log 1 - 1) = 1$$

viii). $\int_{\pi/2}^{3\pi/4} \sin x \, dx = \left[-\cos x \right]_{\pi/2}^{3\pi/4}$

$$= -\left(-\frac{1}{2}\sqrt{2} - 0 \right) = \frac{1}{2}\sqrt{2}$$

(viii). $\int_{-2}^{2} \frac{dx}{x^2+4} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2}$

$$= \frac{1}{2}\left[\frac{1}{4}\pi - \left(-\frac{1}{4}\pi\right)\right] = \frac{1}{4}\pi .$$

7.10 SELF ASSESSMENT QUESTIONS :

1. Integrate the following functions wrt x

(i) x^5 (ii) $\frac{1}{\sqrt{x^3}}$ (iii) $(3x-4)^{2/3}$

(iv) $\frac{\cos x}{1+\cos x}$ (v) $\cos^2 x$, (vi) $\sin^4 x$

2. Evaluate the following integrals :

(i) $\int \frac{x^3\, dx}{1+x^8}$ (ii) $\int x\sqrt{2x+3}\, dx$

(iii) $\int x^2 \sin x^3\, dx$, (iv) $\int \frac{1}{x \log x}\, dx$

3. Evaluate the following integrals :

(i) $\int x \cos x\, dx$ (ii) $\int x^2 e^x\, dx$

(iii) $\int x.\log 2x\, dx$ (iv) $\int x^2 \cos^2 x\, dx$

(v) $\int x \cos^{-1} x\, dx$ (vi) $\int \sin^{-1} x\, dx$

4. Evaluate the following integrals :

(i) $\int \frac{x+7}{(x+4)(x-2)}\, dx$ (ii) $\int \frac{x^2+2x+8}{(x-1)(x-2)}\, dx$

(iii) $\int \frac{1}{(x-3)(x+2)(x-1)}\, dx$ (iv) $\int \frac{3x+1}{(x+3)(x-1)^2}\, dx$

(v) $\int \frac{dx}{\sin x\,(3+2\cos x)}$

5. Evaluate the following :

(i) $\int_0^{\pi/2} \sin dx$ (ii) $\int_0^1 \frac{dx}{1+x^2}$ (iii) $\int_0^2 \sqrt{6x+4}\, dx$ (iv) $\int_0^{\pi/2} \frac{dx}{1+\cos x}$

7.11 KEY WORDS :

Indefinite integral, Integration by parts, Partial fractions, Substitution, Definite integral.

7.12 SUGGESTED READINGS :

1. Grewal, B.S. - Engineering Mathematics

2. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.

3. Ramana, B.V. - Higher Engineering Mathematics.

4. Aggarwal, R.S. - Modern Approach of Mathematics.

Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**	
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No.**	**:**	**08**
Lesson	**:**	**Differential Equations**			

8.0 OBJECTIVES :

In this lesson, you will be able to understand

* to concept of a differential equation & its formation.
* solution of differential equations of first order & first degree by standard methods such as variable seperable, Homogeneous equations, linear equations.

8.1 INTRODUCTION :

A **differential equation** is an equation involving differentials or differential co-efficients, i.e., an equation which expresses a relation between the independent, dependent variables & their derivatives is called a differential eduation. Mathematically , a differential equation can be expressed by the relation

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{d^nx}\right) = 0$$

Here differential equation of first order & first degree can be expressed as $f\left(x, y\frac{dy}{dx}\right) = 0$

i.e., Thus $\frac{dy}{dx} = x^2 - 1$ ...(1)

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + y = 0 \quad ...(2)$$

$$(x + y^2 - 3y)\, dx = (x^2 + 3x + y)\, dy \quad ...(3)$$

$$y = x\frac{dy}{dx} + c\frac{dy}{dx} \quad ...(4)$$

$$\left(\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2}\cdot\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^3 = 0 \quad ...(5)$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = k \cdot \frac{d^2y}{dx^2} \quad ...(6)$$

are all differential equations.

8.2 SOME DEFINITIONS :

(*i*) Differential equations which involve only one independent variable and the differential co-efficients with respect to it are called **ordinary differential equations.**

Here, equations (1) to (6) are all ordinary differential equations.

(*ii*) The **order** of a differential equation is the order of the highest order deriative occurring in the differential equation.

Thus equations (1), (3) and (4) are of first order; equations (2) and (6) are of the second order while equation (5) is of the third order.

(*iii*) The **degree** of a differential equation is the degree of the highest order derivative which occurs in it provided the equation has been made free of the radicals and fractions as far as the derivatives are concerned.

Thus, equations (1), (2), (3) and (5) are of the first degree.

Equation (4) & (6) are of the second degree, can be rewritten as

$$y\frac{dy}{dx} = x\left(\frac{dy}{dx}\right)^2 + c$$

$$\left[1 + \left(\frac{dy}{dx}\right)^2\right]^3 = k^2\left(\frac{d^2y}{dx^2}\right)^2$$

(*iv*) **Solution of a Differential Equation :** A solution (or integral) of a differential equation is a relation, free from derivatives, between the variables which satisfies the given equation.

The **general** (or **complete) solution** of a different equation is that in which the number of independent arbitrary constants is equal to the order of the differential equation.

Thus, $y = c_1 \cos x + c_2 \sin x$ (involving two arbitrary constants c_1, c_2) is general solution of the differential equation $\frac{d^2y}{dx^2} + y = 0$ of second order.

A **particular solution** of a differential equation is that which is obtained from its general solution by giving particular values to the arbitrary constants.

For example, $y = c_1e^x + c_2e^{-x}$ is the general solution of the differential equation $\frac{d^2y}{dx^2} - y = 0$, whereas $y = e^x - e^{-x}$ or $y = e^x$ are its particular solutions.

The solution of a differential equation of nth order is its particular solution if it contains less than n arbitrary constants.

8.3 FORMATION OF A DIFFERENTIAL EQUATION :

Differential equations are formed by elimination of arbitrary constants. To eliminate two arbitrary constants, we require two more equations besides the given relation, leading us to second order derivatives and hence a differential equation of the second order. Elimination of n arbitrary constants leads us to a differential equation of the nth order.

Example 1. Form the differential equation from the equation

$$y = e^x (A \cos x + B \sin x) \text{............}(1)$$

Solution : There are two arbitrary constants A and B in equation (1) .

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = e^x (A \cos x + B \sin x) + e^x (-A \sin x + B \cos x)$$

$$= y + e^x (-A \sin x + B \cos x) \qquad \text{.........}(2)$$

Differentiating again w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A\sin x + B\cos x) + e^x(-A\cos x - B\sin x)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y \qquad \text{[Using (1) and (2)]}$$

or $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0$$

which is the required differential equation.

Example 2 *(a) Form the differential equation of all circles of radius a.*
(b) By eliminating the constants A & B, obtain the differential equation of curve $y = Ae^x + Be^{-x}$

Solution. (a) The equation of any circle of radius a is $(x - h)^2 + (y - k)^2 = a^2$ ...(1)

where (h, k) the coordinates of the centre are arbitrary.

Differentiating (1) w.r.t.x, we have, $2(x-h) + 2(y-k)\frac{dy}{dx} = 0$

or $$(x-h) + (y-k)\frac{dy}{dx} = 0 \qquad ...(2)$$

Differentiating again, we have $1 + (y-k)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ ...(3)

From (3), $$y-k = -\frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

and From (2), $$x-h = -(y-k)\frac{dy}{dx} = \frac{\frac{dy}{dx}\left[1 + \left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}$$

Substituting the values of (x-h) and (y-k) in (1), we get

$$\frac{\left(\frac{dy}{dx}\right)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}+\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2}=a^2$$

or $\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[\left(\frac{dy}{dx}\right)^2+1\right]=a^2\left(\frac{d^2y}{dx^2}\right)^2$ or $\left[1+\left(\frac{dy}{dx}\right)^2\right]^3=a^2\left(\frac{d^2y}{dx^2}\right)^2$

which is the required differential equation.

b) We have $y= Ae^{x}+Be^{-x}$ -------------(1)

Differentiating (1) wrt x, we get

$\frac{dy}{dx}= Ae^{x}- Be^{-x}$ -------------(2)

Again differentiating (2), we obtain

$\frac{d^2y}{dx^2}= Ae^{x}+Be^{-x}$ -------------(3)

Eliminating A and B from (1), (2), & (3), we get

$\frac{d^2y}{dx^2}= y$, which is the required differential equation.

8.4 SOLUTION OF DIFFERENTIAL EQUATIONS OF THE FIRST ORDER AND FIRST DEGREE :

All differential equations of the first order and first degree cannot be solved. Only those which belong to (or can be reduced to) one of the following categories can be solved by the standard methods.

(*i*) Equations in which variables are separable.

(*ii*) Homogeneous equations

(*iii*) Linear equations.

In the following sections, we consider the solution of the above types of equations

8.4.1 VARIABLES SEPARABLE FORM

If a differential equation of the first order and first degree can be put in the form where *dx* and all terms containing *x* are at one place, also *dy* and all terms containing *y* are at one place, then the variables are said to be separable.

Thus the general form of such an equation is $f(x)dx + \phi(y)\,dy = 0$

Integrating, we get $\int f(x)\,dx + \int \phi(y)\,dy = c$ which is the general solution, *c* being an arbitrary constant.

Example 3. *Solve*

$$x(1+y^2)\,dx + y(1+x^2)\,dy = 0.$$

Sol. The given equation is

$$x(1+y^2)\,dx + y(1+x^2)\,dy = 0.$$

Separating the variables, by dividing by $(1+x^2)(1+y^2)$ throughout

$$\frac{x}{1+x^2}\,dx + \frac{y}{1+y^2}\,dy = 0$$

Integrating both sides,

$$\int \frac{x}{1+x^2}\,dx + \int \frac{y}{1+y^2}\,dy = c_1$$

$$\frac{1}{2}\int \frac{2x}{1+x^2}\,dx + \frac{1}{2}\int \frac{2y}{1+y^2}\,dy = c_1$$

$$\frac{1}{2}\log(1+x^2) + \frac{1}{2}\log(1+y^2) = c_1$$

$$\frac{1}{2}\log(1+x^2)(1+y^2) = c_1$$

$$\log(1+x^2)(1+y^2) = 2c_1 = \log c.$$

‹ $(1+x^2)(1+y^2) = c$ is required solution.

Note. The constant $2c_1$ is replaced by another constant $\log c$.

Example 4. *Solve* $\left(y - x\dfrac{dy}{dx}\right) = \dfrac{y}{x}$

Sol. The given equation is

$$\left(y - x\frac{dy}{dx}\right) = \frac{y}{x}$$

i.e. $$xy - x^2\frac{dy}{dx} = y$$

or $$y(x-1) = x^2\frac{dy}{dx}$$

i.e. $$\frac{x-1}{x^2}\,dx = \frac{dy}{y} \quad \text{or} \quad \frac{dy}{y} = \frac{x-1}{x^2}\,dx$$

Integrating both sides,

$$\int\frac{dy}{y} = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx + c$$

$$\log y = \log x + \frac{1}{x} + c$$ which is the required solution.

Example 5. *Solve* $$y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$$

Sol. The given equation can be written as $y(1-ay) - (x+a)\dfrac{dy}{dx} = 0$

or $$\frac{dx}{x+a} - \frac{dy}{y(1-ay)} = 0$$

Integrating, we have

$$\int\frac{dx}{x+a} - \int\left(\frac{1}{y} + \frac{a}{1-ay}\right)dy = c$$

$$\Rightarrow \quad \log(x+a) - \left[\log y + a \cdot \frac{\log(1-ay)}{-a}\right] = c$$

$$\Rightarrow \quad \log(x+a) - \log y + \log(1-ay) = \log C, \text{ where } c = \log C$$

$$\Rightarrow \quad \log \frac{(x+a)(1-ay)}{y} = \log C \quad \Rightarrow \quad (x+a)(1-ay) = Cy$$

which is the general solution of the given equation.

Example 6. If $\frac{dy}{dx} = e^{x+y}$ and it is given that for x=1, y=1; find y when x=-1.

Sol. $\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$

Separating the variables

$$e^{-y}\, dy = e^x\, dx$$

Integrating both sides

$$\int e^{-y}\, dy = \int e^x\, dx + c$$

$$-e^{-y} = e^x + c \qquad \ldots(1)$$

When x=1, y=1 (given)

$$\therefore \quad -e^{-1} = e^1 + c. \qquad i.e., c = -\frac{1}{e} - e = -\left(\frac{1+e^2}{e}\right)$$

Putting the value of c in (1), we have

$$-e^{-y} = e^x - \left(\frac{1+e^2}{e}\right) \qquad \ldots(2)$$

In (2), putting x=-1

$$-e^{-y} = e^{-1} - \left(\frac{1+e^2}{e}\right)$$

$$= \frac{1}{e} - \frac{1+e^2}{e} = \frac{1-1-e^2}{e} = -e$$

$-e^{-y} = -e \qquad \therefore \quad e^{-y} = e$

$\Rightarrow \quad y = -1.$

8.4.2 EQUATIONS REDUCIBLE TO VARIABLES SEPARABLE

This type of equation is generally of the form

$$\frac{dy}{dx} = f(ax + by + c) \qquad ...(1)$$

To solve it, put **ax+by+c = z**

$$\therefore \quad a + b\frac{dy}{dx} = \frac{dz}{dx}$$

or $$\frac{dy}{dx} = \frac{\frac{dz}{dx} - a}{b}$$

Thus (1) becomes,

or $$\frac{\frac{dz}{dx} - a}{b} = f(z)$$

or $$\frac{dz}{dx} = a + b\,f(z)$$

or $$\frac{dz}{a+bf(z)} = dx \qquad ....(2)$$

Thus the variables have been separated in (2). Integrating (2) on both sides, we get the required solution.

$$\int \frac{dz}{a+bf(z)} = x + c.$$

Example 7. *Solve* $\frac{dy}{dx} = \sin(x+y)$

Sol. The given differential equation is

$$\frac{dy}{dx} = sin\,(x+y)$$

Put $x+y = z$

$\therefore$ $1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (1) becomes ;

$$\frac{dz}{dx} = 1+ \sin z$$

Separating the variables,

$$\frac{dz}{1+\sin z} = dx$$

Integrating, we have

$$\int \frac{dz}{1+\sin z} = \int dx + c$$

or $$\int \frac{(1-\sin z)}{(1+\sin z)\,(1-\sin z)}\, dz = \int dx + c$$

$$\int \frac{(1-\sin z)}{\cos^2 z}\, dz = x + c$$

$$\int (\sec^2 z - \tan z \sec z)\, dz = x + c$$

$$(\tan z - \sec z) = x + c$$

$$\therefore \quad \tan (x+y) - \sec (x+y) = x + c$$

which is the required solution.

Example 8. *Solve* $(x+2y)\,(dx-dy) = dx+dy.$

Sol. The given equation is

$$(x+2y)\,(dx-dy) = dx+dy$$

This can be written as

$(x+2y-1)dx = (x+2y+1)\ dy$

or $$\frac{dy}{dx} = \frac{x+2y-1}{x+2y+1} \qquad ...(1)$$

put $x+2y=z$ then $1 + 2\frac{dy}{dx} = \frac{dz}{dx}$

or $$\frac{dy}{dx} = \frac{1}{2}\left(\frac{dz}{dx} - 1\right)$$

‹ (1) becomes, $$\frac{1}{2}\left(\frac{dz}{dx} - 1\right) = \frac{z-1}{z+1}$$

or $$\frac{dz}{dx} = 2\left(\frac{z-1}{z+1}\right) + 1$$

$$= \frac{3z-1}{z+1}$$

i.e., $$\frac{z+1}{3z-1}dz = dx$$

or $$\frac{1}{3}\left(\frac{3z-1+4}{3z-1}\right)dz = dx$$

i.e. $$\left(\frac{1}{3} \quad 1 + \frac{4}{3z-1}\right)dz = dx$$

Integrating both sides ;

$$\frac{1}{3}\int\left(1 + \frac{4}{3z-1}\right)dz = \int dx + c$$

$$\frac{1}{3}\left[z + \frac{4}{3}\ \log(3z-1)\right] = x+c$$

or $3z+4\log(3z-1) = 9x+9c$

$\Rightarrow$ $3(x+2y) + 4\log[3(x+2y) - 1] = 9x+k$ where $k=9c$

$$\Rightarrow \quad 6y-6x+4\log(3x+6y-1) = k$$

$$\Rightarrow \quad 3(y-x)+2\log(3x+6y-1) = \frac{k}{2}$$

$$\Rightarrow \quad 3(y-x)+2\log(3x+6y-1) = \lambda \qquad \text{where } \lambda = \frac{k}{2}$$

is the required solution.

Example 9. $(x-y)^2\dfrac{dy}{dx} = a^2.$

Sol. Put $x-y=z$

$$\therefore \quad 1-\frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = 1-\frac{dz}{dx}$$

$\therefore$ The given equation becomes

$$1-\frac{dz}{dx} = \frac{a^2}{z^2} \quad \text{or} \quad \frac{dz}{dx} = \frac{z^2-a^2}{z^2}$$

Separating the variables,

$$\frac{z^2}{z^2-a^2}\,dz = dx \quad \text{or} \quad \left(1+\frac{a^2}{z^2-a^2}\right)dz = dx$$

Integrating both sides;

$$\int\left(1+\frac{a^2}{z^2-a^2}\right)dz = \int dx + c$$

$$z + a^2\,\frac{1}{2a}\log\frac{z-a}{z+a} = x+c$$

$$\Rightarrow \quad x-y+\frac{a}{2}\log\frac{x-y-a}{x-y+a} = x+c$$

$$\Rightarrow \quad \frac{a}{2}\log\frac{x-y-a}{x-y+a} = y+c$$

which is required solution.

8.4.3 HOMOGENOUS EQUATIONS

A differential equation of the form $$\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)} \qquad \text{......... (1)}$$

is called a homogeneous differential equation if $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of the same degree in x and y.

If $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of degree r in x and y, then

$$f_1(x,y) = x^r \phi_1\left(\frac{y}{x}\right) \quad \text{and} \quad f_2(x,y) = x^r \phi_2\left(\frac{y}{x}\right)$$

Equation (1) reduces to $$\frac{dy}{dx} = \frac{\phi_1\left(\frac{y}{x}\right)}{\phi_2\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right) \qquad \text{...(2)}$$

Putting $\frac{y}{x} = v$ *i.e.* $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Equation (2) becomes $$v + x\frac{dv}{dx} = F(v)$$

Separating the variables, $$\frac{dv}{F(v)-v} = \frac{dx}{x}$$

Integrating, we get the solution in terms of v and x. Replacing v by y/x. This gives the required solution.

Example 10. *Solve* $x^2y\,dx-(x^3+y^3)dy = 0.$

Sol. The given equation is

$$x^2y\,dx-(x^3+y^3)dy = 0.$$

$$\frac{dy}{dx} = \frac{x^2y}{x^3+y^3} \qquad \text{.......(1)}$$

This is homogeneous in x and y.

Put $y = vx$

$$< \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{.......(2)}$$

From (1) and (2); we have

$$v + x\frac{dv}{dx} = \frac{vx^3}{x^3+v^3y^3} = \frac{v}{1+v^3}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{-v^4}{1+v^3}$$

Separating the variables,

$$\frac{1+v^3}{v^4}\,dv = -\frac{dx}{x}$$

or

$$\left(\frac{1}{v^4} + \frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating both sides ;

$$\int\left(\frac{1}{v^4} + \frac{1}{v}\right)dv = -\int\frac{dx}{x} + c$$

$$-\frac{1}{3v^3} + \log v = -\log x + c$$

$$-\frac{1}{3}\frac{x^3}{y^3} + \log\frac{y}{x} = -\log x + c$$

$$-\frac{1}{3}\frac{x^3}{y^3} + \log y - \log x + \log x = c$$

$$-\frac{1}{3}\frac{x^3}{y^3} + \log y = c$$

is the required solution.

Example 11. *Solve* $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

Sol. The given equation is $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

$$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \qquad \text{........(1)}$$

This is homogeneous in x and y

Put $y = vx$

$$\therefore \quad \frac{dy}{dx} = v + x\frac{dv}{dx} \qquad \text{........(2)}$$

From (1) and (2), we have

$$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2 + v^2x^2}}{x}$$

$$= v + \sqrt{1+v^2}$$

$$x\frac{dv}{dx} = \sqrt{1+v^2}$$

or separating the variables,

$$\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$$

Integrating both sides,

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x} + c$$

$\sinh^{-1} v = \log x + c$

$= \log x + \log k$ (replacing c by $\log k$)

or $\log (v + \sqrt{1+v^2}) = \log x + \log k$ $[\because \sinh^{-1}x = \log (x + \sqrt{1+x^2})]$

$$v + \sqrt{1+v^2} = kx$$

or $\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = kx$ *i.e.* $y + \sqrt{x^2 + y^2} = kx^2$

which is the required solution.

Example 12. *Solve* $(1+e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

Sol. The equation is

$$(1+e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0. \qquad \text{...(1)}$$

Put $x = vy$ $\therefore$ $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$ ...(2)

Thus (1) and (2) give,

$$v + y\frac{dv}{dy} = \frac{e^v(v-1)}{e^v + 1}$$

$$y\frac{dv}{dy} = \frac{e^v(v-1)}{e^v + 1} - v = \frac{ve^v - e^v - ve^v - v}{1 + e^v}$$

$$= -\frac{v + e^v}{1 + e^v}$$

Separating the variables,

$$\frac{1 + e^v}{v + e^v}\,dv = -\frac{dy}{y}$$

Integrating both sides,

$$\int \frac{1 + e^v}{v + e^v}\,dv = -\int \frac{dy}{y} + c$$

$$\log(v + e^v) = -\log y + \log k$$

or $$\log(v + e^v) = \log\frac{k}{y}$$

$\therefore$ $$v + e^v = \frac{k}{y}$$

or $$\frac{x}{y} + e^{x/y} = \frac{k}{y}$$

i.e. $$x + y\,e^{x/y} = k$$

which is the required solution.

8.4.4 EQUATIONS REDUCIBLE TO HOMOGENEOUS FORM

A differential equation of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \qquad \text{...(1)}$$

can be reduced to the homogeneous form as follows :

Case I. When $\frac{a}{a'} \neq \frac{b}{b'}$

Putting x = X+h, y = Y+k (h, k are constants)

so that dx = dX, dy = dY

Equation (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$$

$$= \frac{aX+bY+(ah+bk+c)}{a'X+b'Y+(a'h+b'k+c')} \qquad \text{...(2)}$$

Choose h and k such that (2) becomes homogeneous

This requires ah+bk+c = 0 and a'h+b'k+c' = 0

so that $\frac{h}{bc'-b'c} = \frac{k}{ca'-c'a} = \frac{1}{ab'-a'b}$ or $h = \frac{bc'-b'c}{ab'-a'b}$, $k = \frac{ca'-c'a}{ab'-a'b}$

since $\frac{a}{a'} \neq \frac{b}{b'}$, $\therefore$ $ab'-a'b \neq 0$

so that h, k are finite.

< Equation (2) becomes $\frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$

which homogeneous in X, Y and can be solved by putting Y = vX.

Case II. When $\frac{a}{a'} = \frac{b}{b'}$, $ab'-a'b = 0$ and the above method fails.

Now $\quad \dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{m}$ (say) so that $a' = ma,\ b' = mb$

Equation (1) becomes $\quad \dfrac{dY}{dX} = \dfrac{(ax+by)+c}{m(ax+by)+c'} = f(ax+by)$

which can be solved by putting $ax + by = t$.

Example 13. *Solve (3y-7x+7) dx + (7y-3x+3) dy = 0.*

Sol. The given equation can be written as

$$\frac{dy}{dx} = -\frac{3y-7x+7}{7y-3x+3} \qquad \left[\text{Case } \frac{a}{a'} \neq \frac{b}{b'}\right] \qquad ...(1)$$

Putting $\quad x = X+h,\ y = Y+k \quad$ (h, k are constants)

so that $\quad dx = dX,\ dy = dY$

Equation (1) becomes $\quad \dfrac{dY}{dX} = -\dfrac{3(Y+k)-7(X+h)+7}{7(Y+k)-3(X+h)+3}$

$$= -\frac{3Y-7X+(-7h+3k+7)}{7Y-3X+(-3h+7k+3)} \qquad ...(2)$$

Now, choosing h, k such that $-7h+3k+7 = 0$ and $-3h+7k+3 = 0$

Solving these equations $h = 1,\ k = 0$.

With these values of h, k equations (2) reduces to $\dfrac{dY}{dX} = -\dfrac{3Y-7X}{7Y-3X} \qquad ...(3)$

Putting $\quad Y=vX$ so that $\dfrac{dY}{dX} = v+X\dfrac{dv}{dX}$

Equation (3) becomes $\quad v+X\dfrac{dv}{dX} = -\dfrac{3vX-7X}{7vX-3X}$

or $\quad X\dfrac{dv}{dX} = \dfrac{7-3v}{7v-3} - v = \dfrac{7-7v^2}{7v-3}$

Separating the variables $\quad \dfrac{7v-3}{1-v^2}\,dv = 7\dfrac{dX}{X}$ or $\left(\dfrac{2}{1-v} - \dfrac{5}{1+v}\right)dv = 7\dfrac{dX}{X}$

Integrating, we get $\quad -2\log(1-v)-5\log(1+v) = 7\log X+c$

or $$7 \log X+2 \log (1-v) + 5 \log (1+v) = -c$$

or $$\log [X^7 (1-v)^2 (1+v)^5] = -c \quad \text{or} \quad X^7 \left(1 - \frac{Y}{X}\right)^2 \left(1 + \frac{Y}{X}\right)^5 = e^{-c}$$

or $$(X-Y)^2 (X+Y)^5 = C, \quad \text{Where } C = e^{-c} \qquad ...(4)$$

Putting $X = x-h = x-1$, $Y = y-k = y$

Equation (4) becomes $(x-y-1)^2 (x+y-1)^5 = C$

which is the required solution.

Example 14. *Solve* $(3y+2x+4)dx-(4x+6y+5)dy = 0$

Sol. The given equation can be written as

$$\frac{dy}{dx} = \frac{(2x+3y)+4}{2(2x+3y)+5} \qquad \left[\text{Case } \frac{a}{a'} = \frac{b}{b'}\right] \qquad ...(1)$$

Putting $2x+3y = t$ so that $2+3\frac{dy}{dx} = \frac{dt}{dx}$

or $$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx} - 2\right)$$

Equation (1) becomes $\frac{1}{3}\left(\frac{dt}{dx} - 2\right) = \frac{t+4}{2t+5}$

or $$\frac{dt}{dx} = \frac{3t+12}{2t+5} + 2 = \frac{7t+22}{2t+5}$$

Separating the variables $\frac{2t+5}{7t+22}\, dt = dx$ or $\left(\frac{2}{7} - \frac{9}{7} \cdot \frac{1}{7t+22}\right) dt = dx$

Integrating both sides $\frac{2}{7}t - \frac{9}{49} \log (7t+22) = x + c$

or $14t-9\log(7t+22) = 49x+49c$

Putting $t = 2x+3y$, we have $14(2x+3y)-9\log(14x+21y+22) = 49x+49c$

or $21x-42y+9\log(14x+21y+22) = -49c$

or $7(x-2y)+3\log(14x+21y+22) = C$

Which is the required equaxtion. $\left(\text{Where } C = -\frac{49}{3}c\right)$

8.4.5 LINEAR DIFFERENTIAL EQUATIONS

A differential equation is said to be linear if the dependent variable and its derivative occur only in the first degree and are not multiplied together.

The general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q \qquad ...(1)$$

Where P and Q are functions of x only or may be constants.

Equations (1) is also known as *Leibnitz's linear equation.*

To solve it, we multiply both sides by $e^{\int P dx}$, by getting

$$\frac{dy}{dx}e^{\int Pdx} + y(e^{\int Pdx}P) = Q\,e^{\int Pdx}$$

or $$\frac{d}{dx}(ye^{\int Pdx}) = Q\,e^{\int Pdx}$$

Integrating both sides, we have $ye^{\int Pdx} = \int Qe^{\int Pdx}\,dx + c$ ....*(2)*

which is required solution.

Some Important Observations about the linear Differential Equation.

1. We observe that the left hand side of the linear differential equation (1) has become a perfect differential of $ye^{\int Pdx}$ after the equation has been multiplied by the factor $e^{\int Pdx}$ This factor is called Integrating factor of the differential equation and is shortly written as I.F.

2. The solution of the linear differential equation $\frac{dy}{dx} + Py = Q$; P and Q being functions of x only is given by

or $\quad ye^{\int Pdx} = \int Q\,(e^{\int Pdx})\,dx + c \quad$ or $\quad y\,(\text{I.F.}) = \int Q\,(\text{I.F})\,dx + c$

3. The co-efficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing the equation by it throughout.

4. Some equations are recognized as linear differential equations if y is treated as the independent variable and x is treated as the dependent variable. Thus,

$\frac{dx}{dy} + Px = Q$, where P and Q are functions of y only, is a linear differential equation. Thus its solution is obtained from (2) by interchanging x and y.

Example 15. *Solve* $\frac{dy}{dx} + y\tan x = \sec x$.

Sol. The given differential equation is

$$\frac{dy}{dx} + y\tan x = \sec x \qquad ...(1)$$

which is a linear differential equation.

Here P = $\tan x$ and Q = $\sec x$ are functions of x only.

Therefore, I.F. = $e^{\int P dx} = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$

Hence the solution of (1) is given by

$$y(e^{\int P dx}) = \int Q\,(e^{\int P dx})\, dx + c$$

i.e. $$y \sec x = \int \sec x \,.\, \sec x \, dx + c$$

$$= \int \sec^2 x \, dx + c$$

$$= \tan x + c$$

or $$y = \frac{\tan x}{\sec x} + \frac{c}{\sec x}$$

$$y = \sin x + c \cos x$$

is the required solution.

Example 16. *Solve* $\dfrac{dy}{dx} = \dfrac{x+y+1}{x+1}$

Sol. The given equation is

$$\frac{dy}{dx} = \frac{x+y+1}{x+1}$$

$$= \frac{x+1+y}{x+1}$$

$$= 1 + \frac{y}{x+1}$$

$$\therefore \quad \frac{dy}{dx} - \frac{1}{x+1} y = 1 \qquad \text{...(1)}$$

which is a linear differential equation.

Here $P = -\dfrac{1}{x+1}$ and Q=1 are functions of x only.

Now $\int P\,dx = \int -\frac{1}{x+1}\,dx = -\log(x+1)$

$\therefore$ $\text{I.F.} = e^{\int P dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = (x+1)^{-1} = \frac{1}{x+1}$

Hence the solution of the differential equation (1) is

$$y\, e^{\int P dx} = \int Q\, (e^{\int P dx})\, dx + c$$

i.e., $$y\left(\frac{1}{x+1}\right) = \int 1\left(\frac{1}{x+1}\right) dx + c$$

$$= \int \frac{dx}{x+1} + c$$

$$= \log(x+1) + c$$

Hence $\frac{y}{x+1} = \log(x+1) + c$, is the required solution.

Example 17. *Solve* $(x+2y^3)\,\frac{dy}{dx} = y$

Sol. The given equation is

$$(x+2y^3)\,\frac{dy}{dx} = y$$

This can be written as,

$$y\,\frac{dx}{dy} = x+2y^3$$

or $$\frac{dx}{dy} - \frac{1}{y}x = 2y^3 \qquad ...(1)$$

which is of the form

$$\frac{dx}{dy} + Px = Q \text{ ; where P and Q are functions of y only.}$$

Here $P = -\frac{1}{y}$, $Q = 2y^2$

$$\text{I.F.} = e^{\int P dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1}$$

$\therefore$ The solution is,

$$xy^{-1} = \int (2y^2)\, y^{-1} dy + c$$

$$= 2\int y dy + c$$

$$= y^2 + c$$

Hence $x = y^3 + cy$, is the required solution.

8.4.6 EQUATIONS REDUCIBLE TO LINEAR DIFFERENTIAL EQUATIONS

Equations which reduce to the Linear differential equations are of the form

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}y^n ;$$

where P and Q are functions of x only. This type of equation is also called the **'Bernoulli Equation'.**

Solution of Bernoualli's Equation

To solve the equation $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}\,y^n$, *where P and Q are functions of x only.*

Sol. The given equation is

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}\,y^n \qquad \text{...(1)}$$

Dividing both sides of (1) by y^n; we have

$$y^{-n}\frac{dy}{dx} + P\,y^{1-n} = Q \qquad \text{..........(2)}$$

Put $\qquad y^{1-n} = z$

$$\therefore \quad (1-n)\,y^{-n}\,\frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad y^{-n}\,\frac{dy}{dx} = \frac{1}{1-n}\,\frac{dz}{dx}$$

Thus (2) now becomes;

$$\frac{1}{1-n}\,\frac{dz}{dx} + Pz = Q$$

or

$$\frac{dz}{dx} + P(1-n)\,.\,z = Q\,(1-n)$$

which reduces to the linear form and hence can be solved.

Example 18. *Solve* $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

Sol. The given equation is $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y \qquad$ ...(1)

Dividing (1) by $\cos^2 y$; we have

$$\sec^2 y\frac{dy}{dx} + \frac{\sin 2y}{\cos^2 y}\,x = x^3$$

i.e.,

$$\sec^2 y\frac{dy}{dx} + 2\,\frac{\sin y}{\cos y}\,x = x^3$$

or

$$\sec^2 y\frac{dy}{dx} + 2\tan y\,.\,x = x^3 \qquad \text{...(2)}$$

Put $\tan y = z$

$$\therefore \quad \sec^2 y\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (2) becomes ;

$$\frac{dz}{dx} + 2x \, . \, z = x^3 \qquad ...(3)$$

which is a linear differential equation in z.

Here $\quad P = 2x, Q = x^3$

$$\text{I.F.} = e^{\int P dx} = e^{\int 2x\, dx} = e^{x^2}$$

Hence the solution of (3) is

$$z e^{\int P dx} = \int Q (e^{\int P dx})\, dx + c$$

$$z e^{x^2} = \int x^3 e^{x^2} dx + c$$

i.e., $\quad \tan y \, . \, e^{x^2} = \int x^3 e^{x^2} dx + c \qquad ...(4)$

$$\text{Now} \int x^3 e^{x^2} dx = \int x \left(x^2 e^{x^2}\right) dx$$

$$= \frac{1}{2} \int 2x \, x^2 \, e^{x^2} dx \qquad ...(5)$$

Put $\quad x^2 = t \quad \therefore \quad 2x\, dx = dt$

$\therefore$ (5) becomes ;

$$\int x^3 e^{x^2} dx = \frac{1}{2} \int t e^t \, dt$$

$$= \frac{1}{2} \left[t e^t - \int 1 \, . \, e^t dt \right], \text{ Integrating by parts}$$

$$= \frac{1}{2} \left[t e^t - e^t \right] = \frac{1}{2} (t-1)\, e^t$$

$$\int x^3 e^{x^2}\,dx = \frac{1}{2}(x^2-1)\,e^{x^2} \qquad \text{...(6)}$$

Hence from (4) and (6); we have

$$\tan y\, e^{x^2} = \tfrac{1}{2}(x^2-1)\, e^{x^2} + c$$

which is required solution.

Example 19. *Solve* $\frac{dy}{dx} + \frac{y}{x} = y^2 \log x.$

Sol. The given equation is

$$\frac{dy}{dx} + \frac{y}{x} = y^2 \log x. \qquad \text{...(1)}$$

Dividing by y^2 throughout ;

$$y^{-2}\frac{dy}{dx} + \frac{1}{x}\, y^{-1} = \log x. \qquad \text{...(2)}$$

Put $y^{-1} = z$

$\therefore$ $$-1\, y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (2) becomes

$$-\frac{dz}{dx} + \frac{z}{x} = \log x$$

or $$\frac{dz}{dx} - \frac{z}{x} = -\log x$$

which is a linear differential equation in z

Here $P = -\frac{1}{x}$, $Q = -\log x$ are functions of x alone. The Integrating factor

$$\text{I.F.} = e^{\int P dx} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1}$$

Hence the solution is

$$z\,x^{-1} = \int(-\log x)\,x^{-1}\,dx + c$$

$$= -\int \frac{\log x}{x}\,dx + c$$

$$= -\frac{(\log x)^2}{2} dx + c$$

i.e., $$y^{-1}\,x^{-1} = -\frac{(\log x)^2}{2} + c$$

or $$(xy)^{-1} = -\frac{(\log x)^2}{2} + c$$

is the required solution.

8.5 SELF ASSESSMENT QUESTIONS :

1. Solve the following differential equations :

(i) $(1+x^2)\,dy = (1+y^2)\,dx$

(ii) $x\sqrt{1+y^2}\,dx + y\sqrt{1+x^2}\,dy = 0$

2. Solve the following :

(i) $\cos(x+y)\,dy = dx$

(ii) $\frac{dy}{dx} + 1 = e^{x+y}$

3. Solve the following :

(i) $\frac{dy}{dx} = \frac{y-x}{y+x}$

(ii) $\frac{dy}{dx} = \frac{y^2-x^2}{2xy}$

4. Solve $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$

5. Solve $\frac{dy}{dx} = \frac{x+y+3}{2x+2y+1}$

6. Solve $\frac{dy}{dx} = \frac{2y+x-1}{2x+4y+3}$

7. Solve $\frac{dy}{dx} + \frac{1}{x} y = x^n$

8. Solve $\frac{dy}{dx} - \frac{1}{x} y = \frac{1}{x} \log x$

9. Solve $x\frac{dy}{dx} + y = x^2 y^4$

10. Solve $(1+x^2)\frac{dy}{dx} = xy-y^2$

11. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 \sin x$

12. Eliminate the arbitrary constants and obtain the differential equations from the following :

i) $y = cx + c^2$

ii) $y = Ae^{3x} + Be^{2x}$

iii) $xy = Ae^{x} + Be^{-x} + x^2$

iv) $y = Ae^{2x} + Be^{-3x} + Ce^{x}$

v) $Ax^2 + By^2 = 1$

vi) $x = A \cos (nt + R)$

vii) All straight lines in a plane.

viii) All circles of radius r whose centres lie on the x-axis.

8.6 KEY WORDS :

Differential equation, Order, Degree, Variable separable, homogenous, Linear differential equation.

8.7 SUGGESTED READINGS :

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3. Ramana, B.V. - Higher Engineering Mathematics.

4. Aggarwal, R.S. - Modern Approach of Mathematics.

E E E

Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No. :**	**09**
Lesson	**:**	**Statistics : Measures of Central Tendency & Dispersion**		

9.0 OBJECTIVES :

The Objectives of this lesson is to study about various measures of central tendency such as arithmetic mean, geometric mean, harmonic mean. median, mode. Next, we shall study about measures of disperson such as range, quartile deviation, mean deviation & standard deviation.

9.1 INTRODUCTION :

Statistics is the science which deals with methods of collecting, classifying, presenting, comparing and interpreting numerical data collected to throw light on any sphere of enquiry.

9.2 VARIABLE (OR VARIATE) :

A quantity which can vary from one individual to another is called a variable or variate, e.g., heights, weights, ages, wages of persons, rainfall records of cities etc.

Quantities which can take any numerical value within a certain range are called **"continous variables"**, e.g., as the child grows, his/her height takes all possible values from 40 cm to 100 cm.

Quantities which are incapable of taking all possible values are called **"discrete variables"**, e.g., the number of children a man can have are positive integers 1,2, 3 etc.(no value between any two consecutive integers).

9.3 FREQUENCY DISTRIBUTIONS :

Consider the marks obtained by 30 students of a class in a sessional test in mathematics are :- 7, 38, 11, 40, 0, 26, 15, 5, 45, 7, 32, 2, 18, 48, 8, 31, 27, 4, 12, 35, 15, 0, 7, 28, 46, 9, 16, 29, 34, 10.

The data does not give any useful information and this is called **"raw data"** or **"ungrouped data".**

If we express the data in ascending or descending order of magnitude, this does not reduce the bulk of the data, But we can condense the data into classes or groups.

Note : 1. Marks are called the variable (x) and the number of students in a class is known as the frequency of the variable.

2. The table displaying the manner in which frequencies are distributed over various classes is called frequency table.

9.4 MEASURES OF CENTRAL TENDENCY OR AVERAGES :

Tabulation arranges facts in a logical order and helps their understanding and comparison. What is desired is a numerical expression which summarises the characteristic of the group. Measuresof central tendency or measures of location (also called Averages) serve this purpose.

According to Professor Bowley, "Averages are Statistical Constants which enable us to comprehend in a single effort the significance of the whole."

There are five types of averages

1. Arithmatic Average or Mean   3. Harmonic Mean   5. Mode

2. Geometric Mean   4. Median

Here, we shall come across the following three types of series :

(a) **Individual Observations :** (i.e. where frequencies are not given)

Form x : $x_1, x_2, x_3, ..., x_n$.

b) **Discrete Series :** It is a series of observations of the form

x : x_1, x^2 x_s ...., x_n

f : f_1, f_2 f $_3$ ...., f_n

c) **Continuous Series :** It is a series of observations of the form

Class Interval : a_1-a_2 a_2-a_3 ..... a_n-a_{n+1}

frequency (f) : f_1 f_2 .... f_n

For the purpose of further calculations in statistical work, the mid-point of each class is taken to represent the class.

Thus, if m_i is the mid-point of the ith class, then $m_i = \frac{a_i + a_{i+1}}{2}$ and the above series takes the form

Mid-value (m) : m_1, m_2 m_3 ...., m_n

Frequency (f) : f_1, f_2 f_3 ...., f_n

The mid-value of the ith class may also be denoted by x_i. Thus, a continous series is reduced to the form of a discrete series. $\bar{x}$

9.5 ARITHMATIC MEAN :

9.5.1 In the case of Individual Observations (i.e., where frequency is not given)

(i) Direct Method : If x : $x_1, x_2, ..., x_n$ then A.M., $\bar{x}$ is given by

$$= \frac{x_1+x_2+.....+x_n}{n} = \frac{1}{n}\Sigma x$$

(ii) Short Cut Method (Shift of origin) : Shifting the origin to an arbitray point 'a', the formula

$$= \frac{1}{n}\Sigma x \quad \text{becomes} \quad -a = \frac{1}{n}\Sigma(x-a)$$

or $= a + \frac{1}{n}\Sigma d_x$ where (d_x= x-a

Thus, $= a + \frac{1}{n}\Sigma d_x$

where a = arbitrary number, called Assumed Mean

$\Sigma d_x = \Sigma(x\text{-}a) = (x_1\text{-}a) + (x_2\text{-}a) + ...+(x_n\text{-}a)$

= sum of the deviations of the variate x from 'a'

and n = number of observations

In the case of Discrete Series

(i) Direct Method : If the frequency distribution is

$x : x_1, x_2, ..., x_n$

$f : f_1, f_2, ...,f_n$; then

$$x\tilde{N} = \frac{f_1x_1+f_2x_2+..... + f_nx_n}{f_1+f_2+...... + f_n} = \frac{\Sigma fx}{N}$$,where $N = f_1+f_2+......+f_n=\Sigma f$

(ii) Short Cut Method (Shift of origin): Shifting the origin to an arbitrary point 'a', the formula

$$= \frac{1}{N}\Sigma fx \text{ becomes } \bar{x} - a = \frac{1}{N}\Sigma f(x\text{-}a)$$

Thus $= a + \frac{1}{N}\Sigma fd_x$, where $d_x = x\text{-}a$, 'a' is called assumed mean

$\Sigma fd_x = \Sigma f(x\text{-}a)$

$= f_1(x\text{-}a) + f_2(x_2\text{-}a) + ... +f_n (x_n\text{-}a)$

= sum of the products of 'f' and the deviation of the corresponding variate from 'a'.

$N = f_1+f_2+.......+f_n = \Sigma f$ (= sum of frequencies).

Note : If the values are given in terms of class intervals, the mid-values of class intervals are considered as x and then the above formulae are applied.

(iii) In the case of Continuous Series having equal class intervals, say of width h, we use (Shift of origin and change of scale) Step Deviation Method.

Let $u = \frac{x-a}{h}$, then $x = a + hu$

$\therefore \quad \sum fx = \sum f(a+hu) = a\sum f + h\sum fu$

Dividing both sides by $N = \sum f$, we get

$$\frac{\sum fx}{N} = a + \frac{h\sum fu}{N} \quad \text{or} \quad = a + h\frac{\sum fu}{N}, \text{ where } u = \frac{x-a}{h}.$$

9.5.2 Weighted Arithmatic Mean. If the variate values are not of equal importance, we may attach to them 'weights' $w_1, w_2, ..., w_n$ as measures of their importance.

The weighted mean is defined as $_n = \frac{w_1x_1 + w_2x_2 + ... + w_nx_n}{w_1 + w_2 + ... + w_n} = \frac{\sum wx}{\sum w}$

Note : Formulae for weighted mean is the same as for frequency distribution only the frequencies are replaced by weights. $\bar{x}$

Example 1. Calculate by Direct as well as by short-cut method, the arithmetic mean of the following distribution :

Marks (x) :	9	10	11	12	13	14	15	16	17	18
Frequency (f) :	1	2	3	6	10	11	7	3	2	1

Sol. Let $a = 13$ be taken as assumed mean and $d = x - a = x - 13$.

We prefer to take the value lying in centre as assumed mean.

x	f	fx	d_i = x-13	$f_i d_i$
9	1	9	-4	-4
10	2	20	-3	-6
11	3	33	-2	-6
12	6	72	-1	-6
13	10	130	0	0
14	11	154	1	11
15	7	105	2	14
16	3	48	3	09
17	2	34	4	08
18	1	18	5	05
Total	$\sum f_i$=46	$\sum f_i x_i$=623		$\sum f_i d_i$ =25

$\bar{x}$ $\bar{d}$

By Direct Method $= \dfrac{\sum f_i x_i}{\sum f_i} = \dfrac{623}{46} = 13.54$

By Short-cut Method $= a+ \quad = a + \dfrac{\sum f_i d_i}{N}$, gives $\bar{x} = 13 + \dfrac{25}{46} = 13.54$

Example.2. Given the following frequence distribution calculate the Mean by using step deviation method.

Monthly wages in Rs. :	50-70	70-90	90-110	110-130	
No. of workers	4	44	38	28	
Monthly Wates :	130-150	150-170	170-190	190-210	210-230
No. of workers	6	8	12	2	2

Sol.

Class	Mid-Value	Frequency	$u = \frac{x - a}{h}$ a = 140, h = 20	fu	
50-70	60	4	-4	-16	
70-90	80	44	-3	-132	
90-110	100	38	-2	-76	
110-130	120	28	-1	-28	
130-150	140	6	0	0	-252
150-170	160	8	1	8	
170-190	180	12	2	24	
190-210	200	2	3	6	
210-230	220	2	4	8	+46
		$N=\sum f=144$			$\sum fu = -206$

Here, take the assumed mean a = 140,   h = 20 (width of the class) and

Using formula   = a+h   , we get

$$= 140 + h \frac{1}{N} \sum fu$$

$$= 140 + 20 \frac{1}{144} (-206)$$

$$= 140 - 28.61 = 111.39 \quad \text{(approximately)}$$

Example3. Find the average marks of a student for the following distribution :

Marks below :	10	20	30	40	50	60	70	80
No. of students:	25	40	60	75	95	125	190	240

Sol.Here the cumulative frequency distribution is given, first let us change it into simple grouped frequency distribution.

Class	Frequency	Mid-pt. x	$u = \frac{x-35}{10}$	fu	
0-10	25	5	-3	-75	
10-20	15	15	-2	-30	
20-30	20	25	-1	-20	
30-40	15	35	0	0	-125
40-50	20	45	1	20	
50-60	30	55	2	60	
60-70	65	65	3	195	
70-80	50	75	4	200	+475
	$\Sigma f = N = 240$			350	

$$= \frac{1}{N} \Sigma fu = \frac{350}{240} = \frac{35}{24}$$

$\therefore$ $= a + hu$ Here $a = 35$, $h = 10$

$$\therefore \quad = 35 + 10\left(\frac{35}{24}\right) = 35 + \frac{175}{12}$$

$= 35 + 14.58 = 49.58$

9.5.3 Properties of Arithmatic Mean

Property 1. The algebraic sum of the deviations of all the variates from their arithmatic mean is zero.

Proof. Let d_x be the diviation of the variate x from the mean $\bar{x}$, then $d_x = x-\bar{x}$

$$\sum fd_x = \sum f(x-\bar{x}) = \sum fx-\bar{x}\sum f$$

$$= N\bar{x}-N\bar{x} = 0 \qquad Œ \quad = \frac{\sum fx}{N}\text{ ,where } N = \sum f$$

Property II (Mean of the composite series or combined distribution)

If $_i$(i = 1, 2, ..., k) be the arithmatic means of k distributions with respective frequencies n_i (i = 1, 2, ..., k), then mean of the whole distribution obtained by combining the k distributions is given by

$$= \frac{n_1 \quad _1+n_2 \quad _2+......+n_k \quad _k}{n_1+n_2+.....+n_k} = \frac{\sum_i n_i \quad _i}{\sum_i n_i}$$

Proof Let $x_{11},x_{12},x_{13}.......,x_{1n_1}$ be the variables of the first distribution, $x_{21}, x_{22},......, x_{2n_2}$ be the variables of the second distribution, and so on. Then by def.

$$_1=\frac{1}{n_1}\ (x_{11}+x_{12}+........x_{1n_1})$$

$$_2=\frac{1}{n_2}\ (x_{21} + x_{22}+........x_{2n_2})$$

$\bar{x}$

....

$$_k=\frac{1}{n_k}\ (x_{k1}+x_{k2}+.......x_{kn_k})$$

The mean of the whole distribution of size $(n_1+n_2+......+n_k)$ is given by

$$= \frac{(x_{11}+x_{12}+.......+x_{1n_1}) +(x_{21}+x_{22}+......+x_{2n_2}) + ...+ (x_{k1} + x_{k2}+ .....+ x_{kn_k})}{n_1+n_2+.....+n_k}$$

$$= \frac{n_1\bar{x}_1+n_2\bar{x}_2+.......+n_k\bar{x}_k}{n_1+n_2+.....+n_k}$$

$$= \frac{\sum_i n_i\bar{x}_i}{\sum_i n_i}$$

Example. 4 The mean annual salary paid to all employees of a company was Rs. 5000. The mean annual salaries paid to male and female employees were Rs. 5200 and Rs. 4200 respectively. Determine the percentage of males and females employed by the company.

Sol. Let p_1 and p_2 represent percentage of males and females respectively.

then,

$p_1+p_2 = 100$ ....(1)

Mean annual salary of all employees ($\bar{x}$) = Rs. 5000

Mean annual salary of all males ($\bar{x}_1$) = Rs. 5200

Mean annual salary of all females ($\bar{x}_2$) = Rs. 4200

Using $\bar{x} = \frac{p_1\bar{x}_1+p_2\bar{x}_2}{p_1+p_2}$, we get

$5000 = \frac{5200p_1 + 4200\,p_2}{100}$ or $52\,p_1+42p_2=5000$

or $26\,p_1+21p_2=2500$ or $26p_1+21(100-p_1) = 2500$ [Using(1)]

or $5\,p_1 = 2500 - 2100 = 400$

$\therefore$ $p_1=80$ and $p_2 = 100-80=20$

Hence the percentage of males and females is 80 and 20 respectively.

9.6 GEOMETRIC MEAN :

(a) Geometric Mean (G.M.) of a individual observations $x_1,x_2,......x_3$ $(x \neq 0)$ is the nth root of their product and is devoted by G.

Thus, $G = (x_1.x_2.\,......x_n)^{1/n}$

Taking logarithms of both sides $\log G = \frac{1}{n}(\log x_1+\log x_2 + ...+ \log x_n) = \frac{1}{n}\sum_{i}^{n} \log x_i$

$$\therefore \qquad G = \text{antilog}\left[\frac{1}{n}\sum_{i}^{n} \log x_i\right]$$

(b) In case of Frequency distribution : If the variate x has values $x_1, x_2, x_3 ....., x_n$ with frequencies $f_1, f_2, ..., ..., f_n$, respectively and $N = \sum_{i}^{n} f_i$, then G.M. is given by

$$G = (x_1^{f_1} x_2^{f_2} ..... x_n^{f_n})^{1/N}$$

Taking logs, $\log G = \frac{1}{N}[f_1 \log x_1 + f_2 \log x_2 + ... + f_n \log x_n] = \frac{1}{N}\sum_{i}^{n} f_i \log x_i$

$$G = \text{Antilog}\left[\frac{1}{N}\sum_{i=1}^{n} f_i \log x_i\right]$$

(c) In the case of continuous frequency distribution, x is taken to be the value corresponding to the mid-points of the class-intervals.

Example 5. Compute the geometric mean from the following data :

Marks	No. of Students
0-10	10
10-20	5
20-30	8
30-40	7
40-50	20

Sol.

Marks	Mid-values (x)	No.of Students (f)	log x	f logx
0-10	5	10	0.6990	6.9900
10-20	15	5	1.1761	5.8805
20-30	25	8	1.3979	11.1832
30-40	35	7	1.5441	10.8087
40-50	45	20	1.6532	33.0640
		50		67.9264

$$\log G = \frac{1}{N} \Sigma f \log x = \frac{67.9264}{50} = 1.3585$$

$$G = \text{antilog } 1.3585 = 22.83$$

9.7 HARMONIC MEAN :

Harmonic mean of a number of observations is the reciprocal of the arithmetic mean of the reciprocals of the given values. Thus, the harmonic mean H of $x_1, x_2, \ldots\ldots\ldots\ldots x_n$ is

$$H = \frac{1}{\frac{1}{n}\sum_{i=1}^{n}\frac{1}{x_i}} = \frac{n}{\frac{1}{x_i}+\frac{1}{x_2}+\ldots\ldots+\frac{1}{x_n}}$$

If $x_1, x_2, \ldots\ldots\ldots, x_n$, where $x_i \neq 0$ have the frequencies $f_1, f_2, \ldots\ldots\ldots, f_n$ respectively, then harmonic mean is given by

$$H = \frac{1}{\frac{1}{N}\sum_{i=1}^{n}\frac{f_i}{x_i}} = \frac{N}{\frac{f_1}{x_i}+\frac{f_2}{x_2}+\ldots\ldots+\frac{f_n}{x_n}} \; ; \quad N = \sum_{i=1}^{n} f_i$$

In the case of class-intervals, x is taken to be the mid-value of the class-interval.

Example 6. Find out the harmonic mean of the following data :

Marks (out of 150)	No.of students
10	2
20	3
40	6
60	5
120	4

Sol.

x	f	$\frac{I}{x}$	$\frac{f}{x}$
10	2	.100	.200
20	3	.050	.150
40	6	.025	.150
60	5	.017	.085
120	4	.008	.032
	20		.617

$$\text{H.M.} = \frac{N}{\Sigma \frac{f}{x}} = \frac{20}{.617} = 32.4$$

Example 7. A man travelled by a car for 3 days. He covered 480 km each day. He drove1st day 10 hours at 48 k.p.h., 2nd day 12 hours at 40 k.p.h, 3rd day 15 hours at 32 k.p.h., what was his average speed?

Sol. $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$

Here the distance covered on each day remains constant, i.e., 480 km., i.e., numerator is constant. ∴ Use Harmonic Mean to find averages speed.

$$\therefore \quad \text{Average speed} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}}$$

$$= \frac{3}{\frac{1}{48} + \frac{1}{40} + \frac{1}{32}} = \frac{3}{\frac{37}{480}}$$

$$= \frac{480 x 3}{37} = 38\frac{34}{37} \quad \text{km. per hour.}$$

9.8 MEDIAN :

It is that value of the variate which divides the total frequency into two equal parts, i.e., the number of items having values less than the median is equal to the number of items having values greater than the median.

9.8.1 Formulae for Calculation of the Median.

(a) **For Individual Series,** If n denotes the total number of items :

(i) Median = vlaue of $\left(\frac{n+1}{2}\right)$ th item, when n is **odd**.

(ii) Median = mean of the values of $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2}+1\right)$ th items,when n is **even.**

(b) **For Descrete Series:** For the cumulative frequency distribution. If $N = \Sigma f_i$ is the total frequency, then the value of the **variate x** whose cumulative frequency is $\frac{N}{2}$ or just greater than $\frac{N}{2}$ shall be the median.

(c) **For Coninuous (Grouped) Series**

$$\text{Median} = l + \frac{h}{f}\left(\frac{N}{2} - C\right)$$

Where (i) l = lower limit of the **median class**, i.e., the class corresponding to cumulative frequency $\frac{N}{2}$

(ii) h = width of the median class

(iii) f = the frequency of the median class.

(iv) C =cumlative frequency of the class preceding the median class.

9.8.2 Partition Values

These are those values of the variate which divide the total frequency into a number of equal parts.

(i) **Median**. Which divides total frequency into two equal parts.

(ii) **Quartiles**. The values which divide the total frequency into four equal parts are called quartiles and are denoted by Q_i (i=1,2,3).

Q_1 is called the first or lower quartile

Q_3 is called the 3rd or upper quartile

Q_2 is the same as median.

(iii) **Deciles**. Which divide the total frequency into ten equal parts and are denoted by D_i (i=1,2.....,9).

(iv) **Percentiles**, Which divide the frequency into 100 equal parts and are denoted by P_i (i=1,2, ..., 99)

These are calculated by the following formulae :

(a) **For Individual Series**.

Q_i = the value of $i\left(\frac{n+1}{4}\right)$th item (i=1, 2,3)

D_i = the value of $i\left(\frac{n+1}{10}\right)$th item (i=1,2,........9)

P_i = the value of $i\left(\frac{n+1}{100}\right)$th item (i=1, 2,.......99)

Where n denotes the total number of items. (See Example 9.Next).

(b) **For Descrete Series**. We use the same formula as above, only 'n' is replaced by N= Σf_i, the total frequency.

(c) For GroupedFrequency Distribution.

$$Q_i = l + \frac{h}{f}\left(\frac{iN}{4} - C\right) \quad (i=1, 2, 3)$$

$$D_i = l + \frac{h}{f}\left(\frac{iN}{10} - C\right) \quad (i = 1, 2,3, ...9)$$

$$P_i = l + \frac{h}{f}\left(\frac{iN}{100} - C\right) \quad (i = 1, 2, 3,.....99)$$

where l = the lower limit of the class in which the particular partition value lies.

h = width of this class.

f = the frequency of this class.

C = the cumulative frequency upto and including the class preceding the class in which the particular partitions value lies.

N = total frequency

Example 8. Calculate the median, the 1st quartile, the 3rd quartile, 7th octile, 4th decile and 15th perecentile of the following series of marks obtained by 10 candidates in examination.

24, 23, 28, 15, 10, 40, 42, 32, 48, 8

Sol. Arranging the data in ascending order, we get

8, 10, 15, 23, 24, 28, 32, 40, 42, 48

Here number of items n = 10 (even)

(i) Median = $\frac{1}{2}$ (value of 5th item +value of 6th item)

$= \frac{1}{2}$ (24+28) = 26 marks

(ii) Q1 = value of $1\left(\frac{n+1}{4}\right)$th item

$= \text{value of } \left(\frac{10+1}{4}\right)$ th = (2.75)th item.

= value of 2nd item

+0.75 (size of 3rd item - size of 2nd item)

= 10+0.75 (15-10)

= 10+0.75 (5) = 10+3.75 = 13.75 marks

Q_3 = Value of $3\left(\frac{n+1}{4}\right)$th item

$= \text{value of } 3\left(\frac{10+1}{4}\right)$ = (8.25)th item

= value of 8th items

+0.25 (value of 9th item - value of 8th item)

$= 40+0.25(2) = 40+0.50 = 40.5$ marks

(iii) Q_7 = Seventh octile = value of $7\left(\frac{n+1}{8}\right)$th item

$= \text{value of } 7\left(\frac{11}{8}\right)\text{th} = \left(\frac{77}{8}\right)$ th item

= value of (9.6)th item

= value of 9th item + 0.6 (Difference of values of 10th and 9th items)

$= 42+0.6(6)$

$= 42+3.6=45.6$ marks.

(iv) D_4 =4th decile = value of $4\left(\frac{n+1}{10}\right)$ th item

$= \text{value of } 4\left(\frac{10+1}{10}\right)=\left(\frac{44}{10}\right) = (4.4)\text{th item}$

= value of 4th item

+0.4 (value of 5th item - value of 4th item)

$= 23+0.4(1) = 23.4$ marks

(v) P_{15} = 15th percentile = value of $15\left(\frac{n+1}{100}\right)$ th item

$= \text{value of } \left(\frac{15\text{x}11}{100}\right) = (1.65)\text{th item}$

= value of 1st item

+(0.65) (value of 2nd - value of 1st)

$= 8+0.65(2) = 8+1.3=9.3$ marks.

Example. 9. The following data relates to the sizes of shoes sold at a store during a given week. Find the median size of the shoes. Also calculate the quartiles, 3rd decile and 85th percentile.

Size of Shoes	Freqency	Size of Shoes	Frequency
5	2	7.5	40
5.5	8	8	25
6	20	8.5	10
6.5	30	9	3
7	70	9.5	1

Sol. We first find a cumulative frequency distribution.

Size of Shoes	Frequency	Cumulative Frequency
5.0	2	2
5.5	8	10
6.0	20	30
6.5	30	60
7.0	70	130
7.5	40	170
8.0	25	195
8.5	10	205
9.0	3	208
9.5	1	209

(1) Median = Size of $\left(\frac{N+1}{2}\right)$ th or $\left(\frac{209+1}{2}\right)$th i.e., 105th pair =7

(2) LowerQuartile = Q_1 = size of $\left(\frac{N+1}{4}\right)$th pair.

=size of $\left(\frac{105}{2}\right)$th = 52.5th pair

= 6.5 (Œ 52th and 53rd pair belongs to size 6.5)

(3) Upper Quartile = Q_3 = size of $3\left(\frac{N+1}{4}\right)$th pair

= size of $\frac{3}{4}$ (210) = (157.5) th pair = 7.5

(4) 3rd Decile = size of $3\left(\frac{N+1}{10}\right)$ i.e., 63rd pair = 7

(5) 85th Percentile = size of $85\left(\frac{N+1}{100}\right)$, i.e., (178.5)th pair = 8.

Example. 10. Find out the median & quartiles from the following data :

Monthly Rent in Rs.	No.of families paying the rent	Monthly Rent in Rs.	No.of Families paying the rent
20-40	6	120-140	15
40-60	9	140-160	10
60-80	11	160-180	8
80-100	14	180-200	7
100-120	20		

Sol. First form the cumulative frequency distribution

Class	Frequency	Cumulative Frequency	
20-40	6	6	
40-60	9	15	
60-80	11	26	-class containing Q_1
80-100	14	40	
100-200	20	60	(MedianClass)
120-140	15	75	
140-160	10	85	
160-180	8	93	
180-200	7	100	

Here N = 100, $\therefore \frac{N}{2} = 50$, and the cumulative frequency (50) belongs to the class (100-120), which therefore is the median class

(i) Median $= l + \frac{h}{f}\left(\frac{N}{2} - C\right)$

$= 100 + \frac{20}{20}(50-40) = 100+10 = 110$ Rs.

(ii) $Q_1 = l + \frac{h}{f} \quad \frac{N}{4} - C\bigg)$

$\frac{N}{4} = 25$, which belongs to the class-interval (60-80)

$= 60 + \frac{20}{11}(25-15)$

$= 60 + \frac{200}{11} = 60+18.2 =$ Rs. 78.2 App.

(iii) $Q_3 = l + \frac{h}{f}\left(\frac{3N}{4} - C\right)$

$\frac{3N}{4} = 75$, which belongs to the class-interval (120-140)

$= 120 + \frac{20}{15}(75-60) = 140$

9.9 MODE :

The mode or the modal value is that value of the variate which occurs most frequently, i.e.,which has the maximum frequency. We shall denote the mode by M_0.

Note. The mode of a distribution may not exist and even if it exist, it may not be unique. A distribution is sold to be Unimodal, Bimodal or Multimodal according as it has unique mode or two modes or more than two models. Here we discuss some methods to estimate mode of a distribution.

9.9.1 For Descrete Series (i) (Method of Inspection). Arrange the data in the form of an ARRAY and see which variate occurs the maximum numbers of times.

(ii) Method of Grouping. Explained in Example No.13

9.9.2 For Grouped Distribution

$$\textbf{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h, \text{ where}$$

(i) l = lower limit of the modal class, (i.e. class having the maximum frequency)

(ii) f_m = the maximum frequency.

(iii) f_1 = the frequency of class preceding the modal class.

(iv) f_2 = the frequency of class succeeding the modal class.

Note : The class with maximum frequency is called **modal class** and above formula is to be used only when the class-intervals are of equal size. In case the class-intervals are unequal size, they must be made equal on the assumption that the frequencies are equally distriubuted throughout the class.

Example 11. Find the mode of the set of numbers.

2, 2, 10, 10, 5, 7, 9, 9, 11, 13, 18, 9

Sol. Arranging the series in ascending order,we get

2, 2, 5, 7, 9, 9, 9, 10, 10, 11, 13, 18, so the variate 9 occurs the maximum number of times, so Mode = 9.

Example 12. Find the mode of the following series :-

Height	Frequency	Height	Frequency
5'6"	1	5'11"	1
5'7"	2	6'0"	2
5'8"	4	6'1"	1
5'9"	3	6'2"	1
5'10"	2	6'3"	1

Sol. (Method of Grouping) :

Height in Inches	Given frequency	Frequency				
	I	II	III	IV	V	VI
66"	1	3 (66"–67")		**7** (66"–68")		
67"	2		**6** (67"–68")		**9** (67"–69")	
68"	**4**	**7** (68"–69")				**9** (68"–70")
69"	3		5 (69"–70")	6 (69"–71")		
70"	2	3 (70"–71")			5 (70"–72")	
71"	1		3 (71"–72")			4 (71"–73")
72"	2	3 (72"–73")		4 (72"–74")		
73"	1		2 (73"–74")		3 (73"–75")	
74"	1	2 (74"–75")				
75"	1					

Method to obtain the above table :

Column I : Write the original given frequencies.

Column II : Sum of the frequencies taken two at a time starting from 1st frequency of Col. 1.

Column III : Sum of frequencies taken two at a timestarting from 2nd frequency of Col. 1.

Column III : Sum of frequencies taken three at a time starting from 2nd frequency of Col. 1.

Column IV : Sum of frequencies taken three at a time starting from 1st frequency of Col. 1

Column V : Sum of frequencies taken three at a time starting from 2nd frequency of Col. 1

Column VI : Sum of frequencies taken three at a time starting from 3rd frequency of Col. 1

So on.

Maximum frequencies in these columns are indicated by **Bold values.** Form the following analysis table.

Columns	Sizes of items having maximum frequency
I	68
II	68, 69
III	67, 68
IV	66, 67,68
V	67, 68,69
VI	68, 69, 70

In this **analysis table**, the item 68 inches occurs maximum number, i.e., 5 times.

$\therefore$ Mode = 68 = 5'8"

Example 13. Calculate the mode of the distribution given below :-

Monthly wages in Rs.	No. of workers
50-70	4
70-90	44
90-110	38
110-130	28
130-150	6
150-170	8
170-190	12
190-210	2
210-230	2

Sol. Maximum frequency is 44 and this corresponds to the class (70-90), which therefore is the Modal class.

$$\therefore \quad \text{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

$$= 70 + \frac{44-4}{88-4-38} \times 20 = 70 + \frac{40}{46} \times 20$$

$$= 70 + \frac{400}{23} = 87.4$$

Hence, Mode is Rs. 87.4

Example 14. Calculate mode from the following data :-

Marks :	0-10	10-20	20-40	40-50	50-70
Frequency :	5	18	42	36	30

Sol . Here the class-intervals are of **Unequal Size**.We shall first make them of equal size 10. Assuming that the frequencies are equally distributed throughout the class.

Marks :	0-10	10-20	20-30	30-40	40-50	50-60	60-70
Frequency :	5	18	21	21	36	15	15

Now the maximum frequency is 36 which corresponds to the class 40-50, which therefore is modal class.

Using the formula.

$$\text{Mode} = l + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times h$$

Here $l = 40$, $f_m = 36$, $f_1 = 21$, $f_2 = 15$, $h = 10$

$$\text{Mode} = 40 + \frac{36-21}{72-21-15} \times 10$$

$$= 40 + \frac{150}{36} = 40 + 4.17$$

$$= 44.17$$

9.9.3 Estimating the Mode from the Means and the Median

(i) For a symmetrical distribution, mean, median and mode coincide.

(ii) For moderately asymmetrical distribution they are connected by the empirical relation.

Mean - Mode = 3 (Mean - Median)

9.10 DISPERSION OR VARIATION :

Consider the marks of two groups of studentds :

Group A : 57, 58, 62, 63, 64, 64, 65, 66, 70, 71

Group B : 5, 32, 50, 55, 60, 68, 80, 90, 100, 100

Here the mean marks for both the groups the groups A and B is the same. i.e., 64. Also the median for both groups is the same, i.e. 64. Yet the marks of the studends in the two groups are quite different in the sense that the marks in group A are **closer** to the mean marks 64, whereas the marks in group B are very much **scattered** from the mean marks.

Thus a measure of central tendency alone is not suffcient to give a complete idea of the distribution and therefore to draw valid conclusions from the distribution we need some additional measures. One such measure is **Dispersion** and literally dispersion means **"Scatteredness"**. The study of dispersion enables us to know whether the distribution is homogeneous (as the marks of group A) or the distribution is non-homogeneous (as the marks of group B).

Definition : The extent to which numerical data tend to spread about the average value is called dispersion or variation of the data.

The following are the most commonly used **measures of dispersion** :

a) Range

b) Quartile deviation or semi-inter-quartile range.

c) Average (or mean) deviation.

d) Standard deviation.

9.10.1 Range : Range is the difference between the extreme values of the variate.

Range = L-S, where L = Largest and S = Smallest

Co-efficient of Range = $\frac{L-S}{L+S}$

It is easily understood and computed. But it suffers from the drawback that it depends exclusively on the two extreme values. It is not a reliable measure of dispersion.

9.10.2 Quartile Deviation. The difference between the upper and lower quartiles i.e., Q_3-Q_1 is known as the **inter quartile range** and half of it i.e., ½(Q_3-Q_1), is called the **semi-inter-quartile range** or the **quartile deviation**.

Quartile Deviation = $\frac{1}{2}$ (Q_3-Q_1)

Co-efficient of Quartilc Deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$

Example 15. Calculate the quartile deviation of the marks of 39 students in Statistics given below :

Marks :	0-5	5-10	10-15	15-20	20-25	25-30
No. of Students :	4	6	8	12	7	2

Sol. Cumulative frequency table is given below :

Marks	No. of students (f)	c.f.
0-5	4	4
5-10	6	10
10-15	8	18
15-20	12	30
20-25	7	37
25-30	2	39

Here $N = \sum f = 39$

$$\frac{N}{4} = 9.75 \qquad \therefore \text{ Class of } Q_1 \text{ is 5-10}$$

$$Q_1 = l + \frac{h}{f}\left(\frac{N}{4} - C\right) = 5 + \frac{5}{6}(9.75-4) = 5 + \frac{5x5.75}{6} = 9.79$$

$$\frac{3N}{4} = 29.25 \qquad \therefore \text{ Class of } Q_3 \text{ is 15-20}$$

$$Q_3 = l + \frac{h}{f}\left(\frac{3N}{4} - C\right) = 15 + \frac{5}{12}(29.25-18) = 15 + \frac{5x11.25}{12} = 19.69$$

Quartile deviation = $\frac{1}{2}(Q_3 - Q_1) = \frac{1}{2}(19.69 - 9.79) = \frac{1}{2}x9.90 = 4.95$ marks.

9.10.3 Average Deviation or Mean Deviation

If $x_1, x_2, x_3, \ldots, x_n$, occur $f_1, f_2, f_3, \ldots, f_n$ times respectively and $N = \sum_{i=1}^{n} f_i$ the mean deviation from the average A (usually mean or median) is given by

$$\text{Mean deviation} = \frac{1}{N} \sum_{i=1}^{n} f_i |x_i - A|,$$

where $|x_i - A|$ represents the modulus or the absolute value of the deviation $(x_i - A)$.

Since the mean deviation is based on all the values of the variate, it is a better measure of dispersion than range or quartile deviation.

$$\text{Co-efficient of Mean Deviation} = \frac{\text{Mean Deviation}}{\text{Average from which it is calculated}}$$

Example 16. Find the mean deviation from the median of the following frequency distribution.

Marks :	0-10	10-20	20-30	30-40	40-50
No. of Students :	5	8	15	16	6

Sol.

Mid-value	f	c.f.	$\lvert x-M_d \rvert$	$f \cdot \lvert x-M_d \rvert$
5	5	5	23	115
15	8	13	13	104
25	15	28	3	45
35	16	44	7	112
45	6	50	17	102
	50			478

$\frac{N}{2} = 25 \therefore$ Median class corresponds to c.f. 28 i.e., median class is 20-30

$$\text{Median } M = l + \frac{h}{f}\left(\frac{N}{2} - C\right) = 20 + \frac{10}{15}(25-13) = 20+8 = 28$$

Mean deviation from median $= \frac{1}{N} \sum f|x-M| = \frac{478}{50} = 9.56$ marks.

9.10.4 Standard Deviation

Root mean square Deviation. The root-mean square deviation denoted by 's', is defined as the positive square root of the mean of the squares of the deviations from an arbitrary origin A. Thus

$$s = +\sqrt{\frac{1}{N} \sum f_i (x_i - A)^2}$$

When the deviations are taken from the mean $\bar{x}$, the root-mean square deviation is called the **"standard deviation"** and is denoted by σ. Thus

$$\sigma = \sqrt{\frac{1}{N} \sum f_i (x_i - \bar{x})^2}$$

Note : The square of the standard deviation σ^2 is called **variance**.

Short-cut methods for calculating Standard Deviation (σ)

(i) **Direct Method**

$$\sigma = \frac{1}{N}\sum f_i(x_i-\bar{x})^2$$

$$\sigma^2 = \frac{1}{N}\sum f_i(x_i^2-2x_i\bar{x}+\ ^2) = \frac{1}{N}\sum f_i x_i^2-2\ \ .\frac{1}{N}\sum f_i x_i+\ ^2\frac{1}{N}\sum f_i$$

$$= \frac{1}{N}\sum f_i x_i^2-2\ \ .\ \ +\ ^2\frac{1}{N}\ N = \frac{1}{N}\sum f_i x_i^2-\ ^2$$

Therefore, $\sigma = \sqrt{\frac{1}{N}\sum f_i x_i^2-\ ^2}$

(ii) Change of Origin

Let the origin be shifted to an arbitrary point 'a'. Let $d = x-a$ denote the deviation of variate x from the new origin.

$d = x-a$ L $\bar{d} = \ \ -a$

$\therefore$ $d-\ \ = x-\bar{x}$

$$\sigma_x = \sqrt{\frac{1}{N}\sum f(x-\ \ )^2} = \sqrt{\frac{1}{N}\sum f(d-\ \ )^2} = \sigma_d$$

S.D. remains unchanged by shift of origin.

$$\sigma_x = \sigma_d = \sqrt{\frac{1}{N}\sum fd^2-\left(\frac{1}{N}\sum fd\right)^2}$$

Note : In the case of series of individual observations, if the mean is a whole number, take $a = \bar{x}$. In the case of discrete series,when the values of x are not equidistant, take 'a' somewhere in the middle of the x-series.

(iii) Shift of origin and change of scale(Step Deviation Method)

Let the origin be shifted to an arbitrary point 'a'. Let the new scale be $\frac{1}{h}$ times the original scale.

Let $u = \frac{x-a}{h}$, then $hu = x-a$

$h\bar{u} = \ \ -a$ $\therefore$ $h(u-\ \ ) = x-$

$$\sigma_x = \sqrt{\frac{1}{N}\Sigma f(x-\bar{x})^2} = \sqrt{\frac{1}{N}\Sigma fh^2(u-\ )^2} = h\sqrt{\frac{1}{N}\Sigma f(u-\ )^2} = h\,\sigma_u$$

which is independent of a but not h. Hence S.D.is independent of change of origin but not of change of scale.

$$\sigma_x = h\,\sigma_u = h \quad \frac{1}{N}\Sigma fu^2 - \left(\frac{1}{N}\Sigma fu\right)^2$$

Note : In the case of Dicrete Series, when the values of x are equidistant at intervals of h or in the case of continuous series having equal class intervals of width h, use Step Deviation Method.

Relations between measures of dispersion

Mean Deviation $= \frac{4}{5}$(standard deviation) $= \frac{4}{5}\sigma$

Semi-interquartile range $= \frac{2}{3}$(standard deviation) $= \frac{2}{3}\sigma$

9.10.5 CO-EFFICIENT OF DISPERSION

Whenever we want to compare the v~~ariability of two series~~ which differ widely in their averages or which are measured in different units, we calculate the co-efficients of dispersion,which being ratios, are numbers independent of the units of measurement.The co-efficients of dispersion (C.D.) based on different measures of dispersion are as follows :

a) Based on range : C.D. $= \frac{x_{max}-x_{min}}{x_{max}+x_{min}}$

b) Based on quartile deviation : C.D. $= \frac{Q_3-Q_1}{Q_3+Q_1}$

c) Based on mean deviation : C.D. $= \frac{\text{mean deviation}}{\text{average from which it is calculated}}$

d) Based on standard deviation : C.D. $= \frac{\text{S.D.}}{\text{Mean}} = \frac{\sigma}{\ }$

"Co-efficient of variation". It is the percentage variation in the mean, standard deviation being considered as the total variation in the mean.

$$C.V. = \frac{\sigma}{\bar{x}} \times 100$$

Example 17. Find the mean and standard deviation of the following

Series	Frequency
15-20	2
20-25	5
25-30	8
30-35	11
35-40	15
40-45	20
45-50	20
50-55	17
55-60	16
60-65	13
65-70	11
70-75	5

Sol.

Mid-values x	f	$u = \frac{x-47.5}{5}$	fu	fu^2
17.5	2	-6	-12	72
22.5	5	-5	-25	125
27.5	8	-4	-32	128
32.5	11	-3	-33	99
37.5	15	-2	-30	60
42.5	20	-1	-20	20
47.5	20	0	0	0
52.5	17	1	17	17
57.5	16	2	32	64
62.5	13	3	39	117
67.5	11	4	44	176
72.5	5	5	25	125
	N =143		5	1003

$$= a+h\frac{\Sigma fu}{N} = 47.5 + 5 \times \frac{5}{143} = 47.7$$

$$\sigma_x = h\ \sigma_u = h\sqrt{\frac{1}{N}\Sigma fu^2 - \left(\frac{\Sigma fu}{N}\right)^2} = 5\sqrt{\frac{1003}{143} - \left(\frac{5}{143}\right)^2} = 13.25$$

Example 18. Goals scored by two teams A and B in a football season were as follows:

No.of goals scored in a match	No.of matches A	No.of matches B
0	27	17
1	9	9
2	8	6
3	5	5
4	4	3

Find out which team is more consistent.

Sol. Calculation of Co-efficient of variation for the team A :

No.of goals scored (x)	No.of matches (f) $\bar{x}$	d_x=x-2	fd_x	fd_x^2
0	27	-2	-54	108
1	9	-1	-9	9
2	8	0	0	0
3	5	1	5	1
4	4	2	8	56
	N= 53		-50	138

$$= a+h\frac{\Sigma fd_x}{N} = 2+\frac{-50}{53} = 2 - 0.94 = 1.06$$

$$\sigma = \sqrt{\frac{1}{N}\Sigma fd_x^2 \left(\frac{\Sigma fd_x}{N}\right)^2} = \frac{138}{53} - \left(\frac{-50}{53}\right)^2 = 1.31$$

Co-efficient of variation for the team A = $\frac{\sigma}{\bar{x}}$ x 100 = $\frac{1.31x100}{1.06}$ = 123.6

Calculation of co-efficient of variation for the team B :

No.of goals scored (x)	No.of matches (f)	d_x=x-2	fd_x	fd_x^2
0	17	-2	-34	68
1	9	-1	-9	9
2	6	0	0	0
3	5	1	5	5
4	3	2	6	12
	N = 40		-32	94

$$\bar{x} = a+\frac{\Sigma fd_x}{N} = 2 - \frac{32}{40} = 2 - 0.8 = 1.2$$

$$\sigma = \sqrt{\frac{1}{N}\Sigma fd_x^2 - \left(\frac{\Sigma fd_x}{N}\right)^2} = \sqrt{\frac{94}{40} - \left(\frac{-32}{40}\right)^2} = 1.3$$

Co-efficient of variation for the team B = $\frac{\sigma}{\bar{x}} \times 100 = \frac{1.3x100}{1.2}$ = 108.3

Since the coefficient of variation is less for the team B, hence team B is more consistent.

Result : The standard deviations of two series containing n_1 and n_2 members are σ_1 and σ_2 respectively, being measured from their respective means $_1$ and $_2$. If the two series are grouped together as one series of (n_1+n_2) members, then the standard deviation of this series, measured from its mean , is given by

$$\sigma^2 = \frac{n_1\sigma_1^2+n_2\sigma_2^2}{n_1+n_2} + \frac{n_1n_2}{(n_1+n_2)^2}(x\tilde{N}_1-x\tilde{N}_2)^2.$$

Example 19. The first of the two samples has 100 items with mean 15 and standard deviation 3. If the whole group has 250 items with mean 15.6 and stardard deviation 13.44, find the standard deviation of the second group.

Sol. Here $n_1=100,\ \bar{x}_1=15,\ \sigma=3$

$n = n_1+n_2=250,\quad = 15.6,\quad \sigma = \quad 13.44$

$n_2 = 250-100 = 150$

Using $x\tilde{N} = \dfrac{n_1\bar{x}_1+n_2\ _2}{n_1+n_2}$, we have

$15.6 = \dfrac{100\ (15) + 150\ (\ _2)}{250}$ or $150\ \ _2=250 \times 15.6 - 1500=2400$

$\therefore\quad _2 = 16$

$d_1 = \ _1 - \ = 15-15.6 = -0.6$

$d_2 = \ _2 - \ = 16-15.6 = 0.4$

The variance of the combined group σ^2 is given by the formula

$$\sigma^2 = \frac{n_1\sigma_1^2+n_2\sigma_2^2}{n_1+n_2} + \frac{n_1d_1^2 + n_2d_2^2}{n_1+n_2}$$

or $\quad (n_1+n_2)\,\sigma^2 = n_1(\sigma_1^2+d_1^2) + n_2\,(\sigma_2^2+d_2^2)$

$250 \times 13.44 = 100\ (9+0.36) + 150\ (\sigma_2^2+0.16)$

or $\quad 150\ \sigma_2^2 = 250 \times 13.44 - 100 \times 9.36 - 150 \times 0.16 = 3360 - 936 - 24 = 2400$

$\sigma_2^2 = 16$. Hence $\sigma_2 = 4$

9.11 SELF ASSESSMENT QUESTIONS :

1. Calculate the arithmetic mean of the following distribution :

x :	5	10	15	20	25	30	35	40
f :	20	12	9	8	7	11	6	2

2. Find mean & median of the following distribution

Class	Frequency
110-120	6
120-130	25
130-140	48
140-150	72
150-160	22
160-170	3

3. Calculate median and lower & upper quartiles, range, fourth decile for the following distribution :

Marks	Frequency
5-10	5
10-15	6
15-20	15
20-25	10
25-30	5
30-35	4
35-40	2

4. Calculate the mode of the following items :

2,4,13,6,8,10,12,10,17,10,21,10,18,27

5. Calculate the mode in the following series

Size :	15	25	35	45	55	65	75
Frequency :	185	77	34	180	136	23	50

6. Calculate mean, median & mode of the following distribution :

Marks :	0-50	50-100	100-150	150-200	200-250	250-300
Frequency :	5	14	40	60	15	12

7. Compute quartile deviation from the following data :

Size :	4.5-7.5	7.5-10.5	10.5-13.5	13.5-16.5	16.5-19.5
Frequency :	14	24	38	20	4

8. Compute standard deviation from the following data :

Values :	1	2	3	4	5	6	7	8	9
Frequency :	92	49	50	82	102	60	35	24	4

9. The following are the scores of two batsman A and B in a series of innings :

A :	12	115	6	73	7	19	119	36	84	29
B :	47	12	16	42	4	51	37	84	13	0

Who is the better score getter & who is more consistent ?

9.12 KEY WORDS :

Measures of central tendency, mean, median, mode, G.M, H.M, Frequency distribution, Quartiles, Measures of dispersion, range, mean deviation, standard deviation, Coefficient of variation.

9.13 SUGGESTED READINGS :

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Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No.**	**: 10**
Lesson	**:**	**Statistics : Probability & Probability Distributions**		

10.0 OBJECTIVES :

The Objective probability is based on certain laws of nature, which are undisputed or on some experiments conducted for the purpose. Here, we study the various definitins of probability, addition & multiplication laws of probability, Baye's theorem . Further probability distributions such as Binomial, Poisson, Normal distributions are discussed.

10.1 INTRODUCTION :

The word probability or chance is very commonly used in every-day life, *e.g.*,

(i) Probably it may rain tomorrow.

(ii) A's chance of failure is very rare.

(iii) It is possible that A may win the election this time.

All the words such as probably, chance, possible convey the same sense of uncertainity of happening the event. However, under some conditions, a degree of certainity can be assigned by a numerical value to the happening of the event.

10.2 SOME DEFINITIONS :

(a) **Random experiment :** Occurrences which can be repeated a number of times, essentially under the same conditions, and whose result cannot be predicted before hand are known as **random experiments.**

For example, rolling of a die, tossing a coin, taking out balls from an urn.

(b) **Trial:** Performing of an experiment is called a trial, *e.g.*,

(i) Tossing of a coin *(ii)* Throwing a die, are the trials.

(c) **Events or Cases:** The outcomes in a trial are called events or cases, *e.g.*,

(i) Getting a head or a tail in tossing a coin is an event.

(ii) When a die is thrown, the outcome of one of the numbers 1-6 is an event.

(d) **Equally Likely Events:** The events are said to be equally likely when there is no reason to expect happening of one event is preference to the other, *e.g.*,

(i) When an unbiased coin is tossed either a head or a tail will appear. Both the cases are equally likely since there is no reason to expect a head or a tail in preference to the other.

(ii) When a card is drawn from a well-shuffled pack, any one of 52 cards may be drawn. All the cases are equally likely.

(e) **Exhaustive Cases:** All the possible outcomes in a trial are said to be exhausted cases, *e.g.*,

(i) When a die is thrown any one of the faces marked 1,2,........,6 may appear upper-most. Therefore there are six exhaustive cases.

(ii) If two cards are drawn at random from a well-shuffled pack of 52 cards, the number of exhaustive cases is $^{52}C_2$.

(f) **Mutually Exclusive or Imcompatible Events:** If the happening of any one of the events in a trial precludes the happening of all others, such events are said to be mutually exclusive or incompatible events, *e.g.*,

(i) When a coin is a tossed either head or tail appears. Head and tail both cannot appear simultaneously. Appearance of head or tail are two mutually exclusive cases.

(ii) When a die is thrown, no two of the faces can appear upper-most

simultaneously. Thus there are six mutually exclusive cases.

(g) **Sample Space :** Out of the several possible outcomes of a random experiment, one and only one can take place in a trial. The set of all these possible outcomes is called the sample space for the particular experiment and is denoted by S.

For example, (i) If a coin is tossed, the possible outcomes are H (Head) and T (Tail).

Thus S = {H, T}

(ii) If a die is tossed the outcomes would be 1,2,3,4,5 or 6 and therefore the sample space is S = { 1,2,3,4,5,6}

(h) **Sample Point :** The elements of S, the sample space, are called sample points. For example, if a coin is tossed and H and T denote 'Head' and 'Tail' respectively, then S = {H, T}.

The two sample points are H and T.

(i) **Finite Sample Space :** If the number of sample points in a sample space is finite, we call it a finite sample space. (In this chapter, we shall deal with finite sample spaces only).

(j) **Event :** Every subset of S, the sample space is called an event.

Since S = S, S itself is an event; called a **certain event**.

Also, ϕ = S, the null set is also an event, called an **impossible event**.

If $e \in$ S, then e is called an **elementary even**t. Every elementary event contains only one sample point.

(k) **Favourable Cases:** The outcomes which entail the happening of an event in a trial are said to be favourable events, *e.g.*,

(i) If two dice are thrown, the number of favourable cases of getting a sum 4 is

three viz. (1,3), (2,2) and (3,1).

(ii) When a card is drawn from a well-shuffled pack, the number of favourable cases of drawing an ace is four viz. ace of heart, ace of diamond, ace of spade and ace of club.

(*l*) **Independent and dependent events:** Two or more events are said to be *independent* if the happening or non-happening of any one does not depend (or is not affected) by the happening or non-happening of any other. Otherwise they are said to be *dependent*.

For example. If a card is drawn from a pack of well shuffled cards and replaced before drawing the second card, the result of the second draw is independent of the first draw. However, if the first card drawn is not replaced, then, the second draw is dependent on the first draw.

10.3 MATHEMATICAL OR CLASSICAL OR PRIORI DEFINITION OF PROBABILITY :

If a trail results in *n exhaustive*, *mutually exclusive* and *equally likely cases* and *m* of them are favourable to the happening of an event *E*, then the probability of happening of *E* is given by

$$p \text{ or } P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n}.$$

Note 1. Since the number of cases favourable to happening of E is *m* and the exhaustive number of cases in *n*, therefore, the number of cases unfavourable to happening of E are *n*-*m*.

Note 2. The probability that the event E will not happen is given by

$$q \text{ or } P(\overline{E}) = \frac{\text{Unfavourable number of cases}}{\text{Exhaustive number of cases}} = \frac{n-m}{n} = 1 - \frac{m}{n} = 1-p$$

$$p + q = 1 \text{ i.e., } P(E) + P(\overline{E}) = 1$$

Obviously, p and q are non-negative and cannot exceed unity, *i.e.*, $0 \le p \le 1$, $0 \le q \le 1$.

Note 3. If P(E) = 1, E is called a *certain event*, *i.e.*, the chance of its happening is cent per cent.

If P(E) = 0, then E is an *impossible event*.

Note 4. If n cases are favourable to E and m cases are favourable to $\overline{E}$(i.e., unfavourable to E,) then exhaustive number of cases = $n+m$.

$$P(E) = \frac{n}{n+m} \quad \text{and} \quad P(\overline{E}) = \frac{m}{n+m}$$

We say that "*odds in favour of E*" are $n{:}m$ and "*odds against E*" are $m : n$.

10.3.1 LIMITATION OF CLASSICAL DEFINITION OF PROBABILTIY

(a) Classical definition of probability assumes that the number of favourable and exhaustive cases is finite. But *when these cases are indefinitely large*, this definition fails to provide any value of probability.

(b) *When the variate is continuous*, the classical definition fails, e.g., if it is desired to find the probability that the height of a student selected at random from a section of post-graduate students lies between 150 cm and 160 cm, we cannot find the number of favourable and exhaustive cases and hence the definition fails.

(c) In case the *events are not equally likely*, the definition of probability stated in section 10.3 fails, e.g., if it is required to find the probability of falling down a ceiling fan running in perfect order, we might say that its probability is ½ since the fan may or may not fall down. This probability is so high that no body will take the risk of sitting under the fan. But this result is an absured one since the cases that fan may or may not fall down are mutually exclusive but not equally likely. Only in rare cases fan in running condition will fall down.

10.4 STATISTICAL (OR EMPIRICAL) DEFINITION OF PROBABILITY

If a trial is repeated a large number of times under essentially homogeneous and identical conditions, the limiting ratio of favourable cases of an event to the total number of trials, is called the probability of happening an event.

Let an experiment be repeated a large number of times keeping the conditions of performing the experiment to be same at each trial.

If in n trials, an event E happens m times, then the probability of happening of E is given by $$p = P(E) = \underset{n \to \infty}{\text{Lt.}} \frac{m}{n}.$$

Example 1. *A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random . Find the probability that they will both be white.*

Sol. Total number of balls = 7+6+5 = 18.

Out of 18 balls, 2 can be drawn in $^{18}C_2$ ways.

$\therefore$ Exhaustive number of cases = $^{18}C_2 = \frac{18 \times 17}{2 \times 1} = 153$

Out of 7 white balls, 2 can be drawn in $^{7}C_2 = \frac{7 \times 6}{2 \times 1} = 21$ ways.

$\therefore$ Favourable number of cases = 21

Probability $= \frac{21}{153} = \frac{7}{51}$.

Example 2. *Four cards are drawn from a pack of cards. Find the probability that (i) all are diamonds, (ii) there is one card of each suit, and*

(iii) there are two spades and two hearts.

Sol. 4 cards can be drawn from a pack of 52 cards in $^{52}C_4$ ways.

$\therefore$ Exhaustive number of cases = $^{52}C_4 = \frac{52 \times 51 \times 50 \times 49}{4 \times 3 \times 2 \times 1} = 270725.$

(i) There are 13 diamonds in the pack and 4 can be drawn out of them in ${}^{13}C_4$ ways.

$\therefore$ Favourable number of cases = ${}^{13}C_4 = \dfrac{13 \times 12 \times 11 \times 10}{4 \times 3 \times 2 \times 1} = 715.$

Required probability = $\dfrac{715}{270725} = \dfrac{143}{54145} = \dfrac{11}{4165}$.

(ii) There are 4 suits, each containing 13 cards.

$\therefore$ Favourable number of cases = ${}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 13 \times 13 \times 13 \times 13.$

Required Probability $= \dfrac{13 \times 13 \times 13 \times 13}{270725} = \dfrac{2197}{20825}$.

(iii) 2 spades out of 13 can be drawn in ${}^{13}C_2$ ways.

2 hearts out of 13 can be drawn in ${}^{13}C_2$ ways.

$\therefore$ Favourable number of cases = ${}^{13}C_2 \times {}^{13}C_2 = 78 \times 78$

Required Probability $= \dfrac{78 \times 78}{270725} = \dfrac{468}{20825}$.

Example 3. *A bag contains 50 tickets numbered 1,2,3, ...,50 of which five are drawn at random and arranged in ascending order of magnitude ($x_1<x_2<x_3<x_4<x_5$). What is the probability that $x_3 = 30$?*

Sol. Exhaustive number of cases ${}^{50}C_5$.

If $x_3 = 30$, then the two tickets with numbers x_1 and x_2 must come out of 29 tickets numbered 1 to 29 and this can be done in ${}^{29}C_2$ ways. The other two tickets with numbers x_4 and x_5 must come out of the 20 tickets number 31 to 50 and this can be done in ${}^{20}C_2$ ways.

$\therefore$ Favourable number of cases = ${}^{29}C_2 \times {}^{20}C_2$.

Required Probability $= \dfrac{{}^{29}C_2 \times {}^{20}C_2}{{}^{50}C_5} = \dfrac{551}{15134}$.

10.5 AXIOMS :

(i) With each event E (*i.e.*, a sample point) is associated a real number between 0 and 1, called the probability of that event and is denoted by P(E). Thus 0 7 P(E) 71.

(ii) The sum of the probabilities of all simple (elementary) events constituting the sample space is 1. Thus P(S) = 1.

(iii) The probability of a compound event (*i.e.*, an event made up of two or more sample events) is the sum of the probabilities of the simple events comprising the compound event.

Thus, if there are *n equally likely* possible outcomes of a random experiment, then the sample space S contains n sample points and the probability associated with each sample point is $\frac{1}{n}$. [By Axiom (*ii*)]

Now, if an event E consists of *m* sample points, then the probability of E is

$$P(E) = \frac{1}{n} + \frac{1}{n} + \ldots\ldots\ldots\ldots\ldots m \text{ times} = \frac{m}{n}$$

$$= \frac{\text{Number of sample points in E}}{\text{Number of sample points in S}} .$$

This closely agrees with the classical definition of probability.

10.5.1 PROBABILITY OF THE IMPOSSIBLE EVENT IS ZERO, i.e., P(ϕ) = 0

Proof. Impossible event contains no sample point. As such, the sample space S and the impossible event ϕ are mutually exclusive.

$$\Rightarrow \quad S \cup \phi = S \quad \Rightarrow \quad P(S \cup \phi) = P(S)$$

$$\Rightarrow \quad P(S) + P(\phi) = P(S) \quad \Rightarrow \quad P(\phi) = 0.$$

10.5.2 PROBABILITY OF THE COMPLEMENTARY EVENT $\overline{A}$ OF A IS GIVEN BY P($\overline{A}$) = 1- P(A)

Proof. $\overline{A}$ and A are disjoint events. Also $A \cup \overline{A} = S$

$$\therefore \quad P(A \cup \overline{A}) = P(S)$$

$$\Rightarrow \quad P(A) + P(\overline{A}) = 1 \text{ Hence } P(\overline{A}) = 1 - P(A).$$

10.5.3 FOR ANY TWO EVENTS $\overline{A}$ AND B, $P(\overline{A} \cap B) = P(B) - P(A \cap B)$

Proof. $\overline{A} \cap B = [p : p \in B \text{ and } p \notin A]$

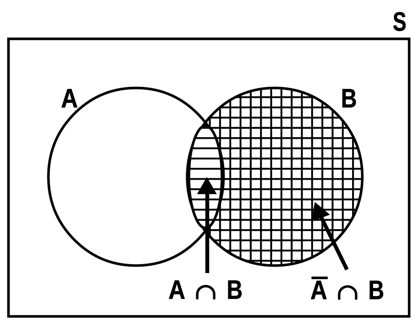
Now $\overline{A} \cap B$ and $A \cap B$ are disjoint sets and

$$(\overline{A} \cap B) \cup (A \cap B) = B$$

$$\Rightarrow \quad P[(\overline{A} \cap B) \cup (A \cap B)] = P(B)$$

$$\Rightarrow \quad P(\overline{A} \cap B) + P(A \cap B) = P(B)$$

$$\Rightarrow \quad P(\overline{A} \cap B) = P(B) - P(A \cap B).$$



Note. Similarly, it can be proved that $P(A \cap \overline{B}) = P(A) - P(A \cap B)$.

IF B _ A, THEN

(i) $P(A \cap \overline{B}) = P(A) - P(B)$ *(ii)* P(B) Ÿ P(A)

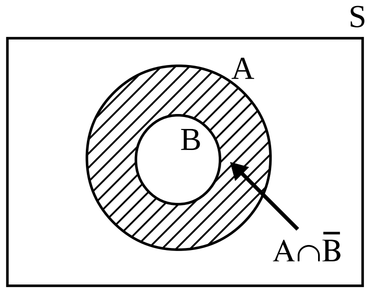
Proof. When B = A, B and $A \cap \overline{B}$ are disjoint and their union is A.

e B 4 (A 3 $\overline{B}$) = A

e P[B 4 (A 3 $\overline{B}$)] = P(A)

$\Rightarrow \quad P(B) + P(A \cap \overline{B}) = P(A)$

$\Rightarrow \quad P(A \cap \overline{B}) = P(A) - P(B) \quad \ldots(1)$



Now, if E is any event,

then $\quad 0 \leq P(E) \leq 1$, *i.e.*, $P(E) \geq 0$

$\therefore \quad P(A \cap \overline{B}) \geq 0 \quad \Rightarrow \quad P(A) - P(B) \geq 0 \quad$ [Using 1]

$\Rightarrow \quad P(B) \leq P(A).$

10.6 ADDITION THEOREM OF PROBABILITIES (OR THEOREM OF TOTAL PROBABILITY) :

Statement. *If A and B are any two events, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e., $\quad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$

Proof. A and $\overline{A} \cap B$ are disjoint sets and their union is $A \cup B$.

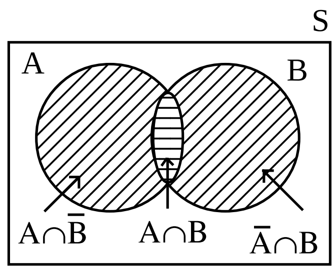
$\Rightarrow \quad A \cup B = A \cup (\overline{A} \cap B)$

$\Rightarrow \quad P(A \cup B) = P(A \cup (\overline{A} \cap B) = P(A) + P(\overline{A} \cap B)$

$= P(A) + [P(\overline{A} \cap B) + P(A \cap B) - P(A \cap B)]$

$= P(A) + P[(\overline{A} \cap B) \cup (A \cap B)] - P(A \cap B)$

[$\because \overline{A} \cap B$ and $A \cap B$ are disjoint]



$= P(A) + P(B) - P(A \cap B)$ $\quad [\because (\bar{A} \cap B) \cup (A \cap B) = B]$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Note 1. If A and B are two mutually disjoint events, then $A \cap B = \phi$, so that $P(A \cap B) = P(\phi) = 0$.

$$\therefore \quad P(A \cup B) = P(A) + P(B).$$

Note 2. $P(A \cup B)$ is also written as $P(A+B)$. Thus, for mutually disjoint events A and B,

$$P(A+B) = P(A) + P(B).$$

$P(A \cap B)$ is also written as $P(AB)$.

10.6.1 IF A, B AND C ARE ANY THREE EVENTS, THEN

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A\cap B\cap C)$$

Or

$$P(A+B+C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC).$$

Using the above Art. for two events, the proof can easily be written in. Further,

IF A_1, A_2, ...,A_n ARE n MUTUALLY EXCLUSIVE EVENTS, THEN THE PROBABILITY OF THE HAPPENING OF ONE OF THEM IS

$$P(A_1 \cup A_2 \cup \ldots\ldots \cup A_n) = P(A_1 + A_2 + \ldots\ldots + A_n) = P(A_1) + P(A_2) + \ldots\ldots + P(A_n)$$

Example 4. *A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace ?*

Sol. Let A = the event of drawing a spade

and B = the event of drawing an ace

A and B are *not* mutually exclusive.

AB = the even of drawing the ace of spades

$$P(A) = \frac{13}{52}, \ P(B) = \frac{4}{52}, \ P(AB) = \frac{1}{52}$$

$$\therefore \quad P(A+B) = P(A) + P(B) - P(AB) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}.$$

10.7 CONDITIONAL PROBABILITY :

The probability of the happening of an event E_1 when another event E_2 is known to have already happened is called *Conditional Probability* and is denoted by $P(E_1/E_2)$.

Mutually Independent Events. An event E_1 is said to be independent of an event E_2 if

$$P(E_1/E_2) = P(E_1).$$

i.e., if the probability of happening of E_1 is independent of the happening of E_2.

10.7.1 MULTIPLICATIVE LAW OF PROBABILITY (OR THEOREM OF COMPOUND PROBABILITY)

The probability of simultaneous occurrence of two events is equal to the probability of one of the events multiplied by the conditional probability of the other, *i.e.*, **for two events A and B**,

$$\mathbf{P(A \cap B) = P(A) \times P(B/A)}$$

where P(B/A) represents the conditional probability of occurrence of B when

the event A has already happened.

Proof. Suppose a trial results in *n* exhaustive, mutually exclusive and equally likely outcomes, m of them being favourable to the happening of the event A.

$\therefore$ Probability of happening of the event A = P(A) = $\dfrac{m}{n}$ (1)

Out of m outcomes favourable to the happening of A, let m_1 be favourable to the happening of the event B.

$\therefore$ Conditional probability of B, given that A has happened = P(B/A) = $\dfrac{m_1}{m}$ (2)

Now out of n exhaustive, mutually exclusive and equally likelly outcomes, m_1 are favourable to the happening of '*A*' and '*B*'.

$\therefore$ Probability of simultaneous occurrence of A and B

$$= P(A \cap B) = \frac{m_1}{n} = \frac{m_1}{m} \times \frac{m}{n} = \frac{m}{n} \times \frac{m_1}{m}$$

$$= P(A) \times P(B/A) \qquad \text{[Using (1) and (2)]}$$

Hence $P(A \cap B) = P(A) \times P(B/A)$.

Note. $P(A \cap B)$ is also written as P(AB)

Thus P(AB) = P(A) x P(B/A).

Cor.1. Interchanging A and B

$$P(BA) = P(B) \times P(A/B)$$

$$P(AB) = P(B) \times P(A/B) \qquad [\text{Œ } B \cap A = A \cap B]$$

Cor. 2. If A and B independent events, then P(B/A) = P(B)

$\therefore$ P(AB) = P(A) x P(B).

Generalisation. If $A_1, A_2, ..., A_n$ are *n* independent events, then

$$P(A_1, A_2 .... A_n) = P(A_1) \text{ x } P(A_2) \text{ x } .... \text{ x } P(A_n).$$

Example 5 *A problem in mechancis is given to three students A, B, C whose chances of solving it are* $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ *respectively. What is the probability that the problem will be solved?*

Sol. The probability of A, B, C solving the problem are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

The probability of A, B, C not solving the problem are $1-\frac{1}{2}, 1-\frac{1}{3}, 1-\frac{1}{4}$ i.e., $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$

The Probability that the problem is not solved by any of them $= \frac{1}{2} \text{ x } \frac{2}{3} \text{ x } \frac{3}{4} = \frac{1}{4}$.

Hence, the probability that the problem is solved by at least one of them $= 1-\frac{1}{4} = \frac{3}{4}$

Example 6. *A can hit a target 4 times in 5 shots; B 3 times 4 shots; C twice in 3 shots. They fire a volley. What is the probability that at least two shots hit ?*

Sol. Probability of A's hitting the target $= \frac{4}{5}$

Probability of B's hitting the target $= \frac{3}{4}$

Probability of C's hitting the target $= \frac{2}{3}$

For atleast two hits, we may have

(i) A, B, C all hit the target , the probabiltiy for which is

$$\frac{4}{5} \text{ x } \frac{3}{4} \text{ x } \frac{2}{3} = \frac{24}{60}.$$

(ii) A, B hit the target and C misses it, the probability for which is

$$\frac{4}{5} \text{ x } \frac{3}{4} \text{ x } \left(1-\frac{2}{3}\right) = \frac{4}{5} \text{ x } \frac{3}{4} \text{ x } \frac{1}{3} = \frac{12}{60}.$$

(iii) A, C hit the target and B misses it, the probability for which is

$$\frac{4}{5} \times \left(1-\frac{3}{4}\right) \times \frac{2}{3} = \frac{4}{5} \times \frac{1}{4} \times \frac{2}{3} = \frac{8}{60}\ .$$

(iv) B, C hit the target and A misses it, the probability for which is

$$\left(1-\frac{4}{5}\right) \times \frac{3}{4} \times \frac{2}{3} = \frac{1}{5} \times \frac{3}{4} \times \frac{2}{3} = \frac{6}{60}\ .$$

Since these are mutually exclusive events, required probability

$$= \frac{24}{60} + \frac{12}{60} + \frac{8}{60} + \frac{6}{60} = \frac{50}{60} = \frac{5}{6}\ .$$

Example 7. *A and B throw alternatively with a single die, A having the first throw. The persn who first throws ace is to win. What are their respective chances of winning?*

Sol. The chance of throwing an ace with a single die = $\frac{1}{6}$

The chance of not throwing an ace with a single die = $1-\frac{1}{6} = \frac{5}{6}$.

If A is to win, he should throw an ace in the first or third or fifth,......., throws.

If B is to win, he should throw an ace in the second or fourth or sixth, ..., throws.

The chances that an ace is thrown in the first, second, third, ..., throws are

$$\frac{1}{6},\ \frac{5}{6}\cdot\frac{1}{6},\ \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6},\ \frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{1}{6}\ ....$$

or $$\frac{1}{6},\ \frac{5}{6}\cdot\frac{1}{6},\ \left(\frac{5}{6}\right)^2\cdot\frac{1}{6},\ \left(\frac{5}{6}\right)^3\cdot\frac{1}{6}\ ,........$$

$\therefore$ A's chance $= \frac{1}{6} + \left(\frac{5}{6}\right)^2\cdot\frac{1}{6} + \left(\frac{5}{6}\right)^4\cdot\frac{1}{6} + ... = \dfrac{\frac{1}{6}}{1-\left(\frac{5}{6}\right)^2}$

$= \frac{6}{11}$

B's chance $= 1-\frac{6}{11} = \frac{5}{11}$.

Sum of an infinite G.P. = $\frac{a}{1-r}$

Example 8. An urn contains 10 white and 3 black balls, while another urn contains 3 white and 5 black balls. Two balls are drawn from the first urn and put into the second urn and then a ball is drawn from the latter. What is the probability that it is a white ball?

Sol. The two balls drawn from the first urn may be

(i) both white *(ii)* both black *(iii)* one white and one black.

Let these events be denoted by A, B, C repectively.

$$P(A) = \frac{{}^{10}C_2}{{}^{13}C_2} = \frac{10 \text{ x } 9}{13 \text{ x } 12} = \frac{15}{26} \quad ;$$

$$P(B) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{3 \text{ x } 2}{13 \text{ x } 12} = \frac{1}{26} \quad ;$$

$$P(C) = \frac{{}^{10}C_1 \text{ x } {}^{3}C_1}{{}^{13}C_2} = \frac{10 \text{ x } 3}{\frac{13 \text{ x } 12}{2 \text{ x } 1}} = \frac{10}{26}$$

When two balls are transferred from first urn to second urn, the second urn will contain.

(i) 5 white and 5 black balls. *(ii)* 3 white and 7 black balls

(iii) 4 white and 6 black balls.

Let W denote the event of drawing a white ball from the second urn in the three cases *(i)*, *(ii)* and *(iii)*.

Now $P(W/A) = \frac{5}{10}$, $P(W/B) = \frac{3}{10}$, $P(W/C) = \frac{4}{10}$

$\therefore$ Reqd. probability $= P(A) \cdot P(W/A) + P(B) \cdot P(W/B) + P(C) \cdot P(W/C)$

$$= \frac{15}{26} \cdot \frac{5}{10} + \frac{1}{26} \cdot \frac{3}{10} + \frac{10}{26} \cdot \frac{4}{10}$$

$$= \frac{75+3+40}{260} = \frac{118}{260} = \frac{59}{130} .$$

10.8 BAYE'S THEOREM :

If $E_1, E_2, ...,E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$, $(i= 1, 2, ..., n)$ of a experiment then for any arbitrary event A of the sample space of the above experiment with $P(A)>0$, we have

$$P(E_i/A) = \frac{P(E_i)\,P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \quad .$$

Proof. Let S be the sample space of the random experiment.

The event $E_1, E_2, ..., E_n$ being exhaustive

$$S = E_1 \cup E_2 \cup ......\cup E_n$$

$$A = A \cap S \qquad [\because A \subset S]$$

$$= A \cap (E_1 \cup E_2 \cup..... \cup E_n)$$

$$=(A \cap E_1) \cup (A \cap E_2) \cup ....\cup (A \cap E_n) \quad \text{[Distributive Law]}$$

$$\Rightarrow \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + \; .... + P(A \cap E_n)$$

$$P(A) = P(E_1)P(A/E_1)+P(E_2)P(A/E_2) + ...+P(E_n)P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A/E_i) \qquad (1)$$

Now $\quad P(A \cap E_i) = P(A)P(E_i /A) = P(E_i)\, P(A/E_i)$

$$P(E_i/A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \qquad \text{[Using (1)]}$$

Example 9. *A bag contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be* *red. Find the probability that it was drawn from bag Y.*

Sol. Let E_1 : the ball is drawn from X

E_2 : the ball is drawn from Y

and A : the ball is red.

We have to find $P(E_2/A)$. By Baye's Theorem,

$$P(E_2/A) = \frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1) + P(E_2)\,P(A/E_2)} \qquad (1)$$

Since the two bags are equally likely to be selected, $P(E_1) = P(E_2) = \frac{1}{2}$

Also $P(A/E_1)$ = P(a red ball is drawn from bag X) = $\frac{3}{5}$

$P(A/E_2)$ = P(a red ball is drawn from bag Y) = $\frac{5}{9}$

$\therefore$ From (1), we have

$$P(E_2/A) = \frac{\frac{1}{2} \times \frac{5}{9}}{\frac{1}{2} \times \frac{3}{5} + \frac{1}{2} \times \frac{5}{9}} = \frac{25}{52}.$$

Example10. *In a bolt factory, machines A, B and C manufacture repectively 25%, 35% and 40% of the total. Of their output 5, 4 and 2 percent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that is was manufactured by machine B?*

Sol. Let E_1, E_2 and E_3 denote the events that a bolt selected at random is manufacture by the machines A, B and C respectively and let H denote the event of its being defective. Then

$$P(E_1) = 0.25,\ P(E_2) = 0.35,\ P(E_3) = 0.40$$

The probability of drawing a defective bolt manufactured by machine A is

$P(H/E_1) = 0.05$

Similarly, $P(H/E_2) = 0.04$ and $P(H/E_3) = 0.02$

By Baye's Theorem, we have

$$P(E_2/H) = \frac{P(E_2)P(H/E_2)}{P(E_1)P(H/E_1) + P(E_2)\ P(H/E_2) + P(E_3)\ P(H/E_3)}$$

$$= \frac{0.35 \times 0.04}{0.25 \times 0.05 + 0.35 \times 0.04 + 0.40 \times 0.02} = \frac{0.0140}{0.0345} = 0.41$$

10.9 PROBABILITY DISTRIBUTIONS :

10.9.1 RANDOM VARIABLE

If a particular value cannot be exactly predicted in advance, the variable is then called a random variable. A random variable is also called *'chance variable'* or *'stochastic variable'*.

Random variables are denoted by capital letters, for example X, Y, Z etc.

10.9.2 Continuous and Discrete Random Variables

A *continuous random variable* is one which can assume any value within an interval, i.e., all values of a continuous scale. For example (i) the weights(in kg) of a group of individuals, (ii) the heights of a group of individuals.

A *discrete random variable* is one which can assume only isolated values. For example,

(i) the number of heads in 4 tosses of a coin is a discrete random variable as it cannot assume values other than 0,1,2,3,4.

(ii) the number of aces in a draw of 2 cards from a well shuffled deck is a random variable as it can take the values 0,1,2 only.

10.9.3 DISCRETE PROBABILITY DISTRIBUTION

Let a random variable X assume values $x_1, x_2, x_3, \ldots\ldots\ldots, x_n$ with probabilities $p_1, p_2, p_3, \ldots\ldots\ldots, p_n$ respectively, where $P(X = x_i) = p_i \geq 0$ for each x_i and $p_1+p_2+p_3+\ldots\ldots+p_n = \sum_{i=1}^{n} p_i = 1$. Then

X	:	x_1,	x_2,	$x_3, \ldots\ldots\ldots, x_n$
P(X)	:	p_1,	p_2,	$p_3, \ldots\ldots\ldots, p_n$

is called the discrete probability distribution for X

10.9.4 MEAN AND VARIANCE OF RANDOM VARIABLES

Let

X :	x_1,	x_2,	$x_3, \ldots\ldots\ldots, x_n$
P(X):	p_1,	p_2,	$p_3, \ldots\ldots\ldots, p_n$

be a discrete probability distribution.

We denote the mean by μ and define $\mu = \dfrac{\Sigma\, p_i x_i}{\Sigma\, p_i} = \Sigma\, p_i x_i \quad (\because \Sigma\, p_i = 1)$

Other names for the mean are *average or expected value E(X).*

We denote the *variance* by σ^2 and define $\sigma^2 = \Sigma\, p_i (x_i - \mu)^2$

If μ is not a whole number, then $\sigma^2 = \Sigma\, p_i x_i^2 - \mu^2$

Standard deviation $\sigma = +\sqrt{\text{Variance}}$.

Example 11. *A random variable X has the following probability function :*

Values of X, x :	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
p(x) :	*0*	*k*	*2k*	*2k*	*3k*	k^2	$2k^2$	$7k^2+k$

(i) Find k, *(ii) Evaluate* $P(X<6)$, $P(x \geq 6)$, $P(3<X \leq 6)$

(iii) Find the minimum value of x so that $P(X \leq x) > \dfrac{1}{2}$.

Sol. *(i)* Since $\sum_{x=1}^{7} p(x) = 1$, we have

$$0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$\Rightarrow \quad 10k^2+9k-1=0 \qquad \Rightarrow \quad (10k-1)(k+1) = 0$$

$$k = \frac{1}{10} \qquad [\because p(x) \geq 0]$$

(ii) $P(X<6) = P(X=0)+P(X=1)+\ldots\ldots\ldots+P(X=5)$

$$=0+k+2k+2k+3k+k^2 = 8k+k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$$

$P(X \geq 6) = P(X=6) + P(X=7)$

$$= 2k^2+7k^2+k = \frac{9}{100} + \frac{1}{10} = \frac{19}{100}$$

$P(3<X \leq 6) = P(X=4) + P(X=5) + P(X=6)$

$$= 3k+k^2+2k^2 = \frac{3}{10} + \frac{3}{100} = \frac{33}{100}$$

(iii) $P(X \leq 1) = k = \frac{1}{10} < \frac{1}{2}$; $\qquad P(X \leq 2) = k+2k = \frac{3}{10} < \frac{1}{2}$

$P(X \leq 3) = k+2k+2k = \frac{5}{10} = \frac{1}{2}$; $\qquad P(X \leq 4) = k+2k+2k+3k = \frac{8}{10} > \frac{1}{2}$

The maximum value of x so that $P(X \leq x) > \frac{1}{2}$ is 4.

10.10 THEORETICAL DISTRIBUTIONS :

There are many types of theoretical frequency distributions but we shall consider only three, which are of great importance :

(i) Binomial Distribution (or Bernoulli's Distribution) ;

(ii) Poission's Distribution ;

(iii) Normal Distribution.

10.11 BINOMIAL PROBABILITY DISTRIBUTION :

Let there be n independent trials in an experiment. Let a random variable X denote the number of successes in these n trials. Let p be the probability of a success and q that of a failure in a single trial so that $p+q=1$. Let the trials be independent and p be constant for every trial.

Let us find the probability of r successes in n trials.

r successes can be obtained in n trials in ${}^nC_r$ ways.

$$P(X=r) = {}^nC_r P(\underbrace{S\ S\ S\ \ldots\ldots S}_{r \text{ times}}\ \ \underbrace{F\ F\ F\ \ldots\ldots F}_{(n-r)\text{times}})$$

$$= {}^nC_r \underbrace{P(S)\ P(S)\ \ldots\ldots P(S)}_{r \text{ factors}}\ \ \underbrace{P(F)\ P(F)\ \ldots\ldots P(F)}_{(n-r)\text{factors}}$$

$$= {}^nC_r \underbrace{p\ p\ p\ \ldots\ldots p}_{r \text{ factors}}\ \ \underbrace{q\ q\ q\ \ldots\ldots q}_{(n-r)\text{factors}}$$

$$= {}^nC_r p^r q^{n-r} \qquad \ldots(1)$$

Hence, $\mathbf{P(X=r) = {}^nC_r\, q^{n-r}\, p^r}$ where $\mathbf{p+q=1}$ **and** $\mathbf{r = 0,1,2,\ldots\ldots n.}$

The distribution(1) is called the *binomial probability distribution* and X is called the *binomial variate*.

Note 1. The successive probabilities P(r) in (1) for $r = 0,1,2, \ldots\ldots,n$ are

$${}^nC_0 q^n,\ {}^nC_1 q^{n-1}p,\ {}^nC_2 q^{n-2}p^2,\ \ldots\ldots\ldots,\ {}^nC_n p^n$$

which are the successive terms of the binomial expansion of $(q+p)^n$. That is why this distribution is called "binomial" distribution.

Note 2. *n* and *p* are called the *parameters* of the distribution.

Note 3. In a binomial distribution :

(i) n, the number of trials is finite.

(ii) each trial has only two possible outcomes usually called success and failure.

(iii) all the trials are independent.

(iv) p (and hence *q*) is constant for all the trials.

10.11.1 RECURRENCE RELATION FOR THE BINOMIAL DISTRIBUTION

In a binomial distribution,

$$P(X= r) = P(r) = {}^nC_r q^{n-r} p^r = \frac{n\,!}{(n-r)\,!\,r\,!}\; q^{n-r}\, p^r$$

$$P(r+1) = {}^nC_{r+1}\, q^{n-r-1}\, p^{r+1} = \frac{n\,!}{(n-r-1)\,!(r+1)\,!} \cdot q^{n-r-1}\, p^{r+1}$$

$$\frac{P(r+1)}{P(r)} = \frac{(n-r)!}{(n-r-1)\,!} \times \frac{r!}{(r+1)\,!} \times \frac{p}{q} = \frac{(n-r) \times (n-r-1)\,!}{(n-r-1)\,!} \times \frac{r!}{(r+1) \times r\,!} \times \frac{p}{q}$$

$$= \frac{n-r}{r+1} \cdot \frac{p}{q}$$

$$\therefore \quad P(r+1) = \frac{n-r}{r+1} \cdot \frac{p}{q}\; P(r)$$

which is the required relation. Applying this formula successively, we can find P(1), P(2), P(3), ..., if P(0) is known.

10.11.2 MEAN AND VARIANCE OF THE BINOMIAL DISTRIBUTION

For the binomial distribution, $P(r) = {}^nC_r q^{n-r} p^r$

$$\text{Mean, } \mu = \sum_{r=0}^{n} rP(r) = \sum_{r=0}^{n} r.\, {}^nC_r q^{n-r} p^r$$

$$= 0+1.\,{}^{n}C_1 q^{n-1} p+2.\,{}^{n}C_2\, q^{n-2}p^2+3.\,{}^{n}C_3 q^{n-3}p^3+.....+n.{}^{n}C_n p^n$$

$$=nq^{n-1}\, p+2.\,\frac{n(n-1)}{2.1}\, q^{n-2}p^2+3.\frac{n(n-1)(n-2)}{3.2.1}\, q^{n-3}p^3+....+np^n$$

$$=nq^{n-1}p + n(n-1)\, q^{n-2}p^2+ \frac{n(n-1)(n-2)}{2.1}\, q^{n-3}p^3+.......+ np^n$$

$$= np\left[q^{n-1}+(n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2.1}\, q^{n-3}p^2+.....+ p^{n-1}\right]$$

$$= np[{}^{n-1}C_0 q^{n-1} + {}^{n-1}C_1 q^{n-2}p + {}^{n-1}C_2 q^{n-3}\, p^2+.....+ {}^{n-1}C_{n-1}p^{n-1}]$$

$$=np(q+p)^{n-1} = np \qquad (\because p+q =1)$$

Hence, the mean of the binomial distribution is "*np*"

$$\text{Variance, } \sigma^2 = \sum_{r=0}^{n} r^2P(r)- \mu^2 = \sum_{r=0}^{n} [r+r(r-1)]\, P(r)-\mu^2$$

$$= \sum_{r=0}^{n} rP(r) + \sum_{r=0}^{n} r(r-1)P(r) -\mu^2 = \mu + \sum_{r=2}^{n} r\,(r-1)\, {}^{n}C_r q^{n-r}p^r- \mu^2$$

(since the contribution due to $r = 0$ and $r =1$ is zero)

$$= \mu +[2.1.{}^{n}C_2 q^{n-2}p^2+3.2.nC_3 q^{n-3}p^3 +.......+n(n-1)\, {}^{n}C_n p^n] -\mu^2$$

$$= \mu + \left[2.1.\frac{n(n-1)}{2.1} q^{n-2}p^2+3.2.\, \frac{n(n-1)(n-2)}{3.2.1}\, q^{n-3}p^3 +.....+n(n-1)p^n \right]- \mu^2$$

$$= \mu +[n(n-1)q^{n-2}p^2+n(n-1)(n-2)q^{n-3}p^3 +.......+n(n-1)p^n]- \mu^2$$

$$= \mu +n(n-1)p^2[q^{n-2}+(n-2)q^{n-3}p +......+ p^{n-2}]- \mu^2$$

$$= \mu +n(n-1)p^2[{}^{n-2}C_o q^{n-2}+{}^{n-2}C_1 q^{n-3}p +..........+{}^{n-2}C_{n-2}p^{n-2}]- \mu^2$$

$$= \mu +n(n-1)p^2(q+p)^{n-2}- \mu^2 = \mu + n(n-1)p^2- \mu^2 \qquad [\because q+p =1]$$

$$= np + n(n-1)p^2-n^2p^2 \qquad [\because \mu = np]$$

$$= np[1+(n-1)p-np] = np[1-p]= nqp.$$

Hence the variance of the binomial distribution is "*nqp*".

Standard deviation of the binomial distribution is $\sqrt{npq}$.

10.11.3 RECURRENCE RELATION FROM MOMENTS OF BINOMIAL DISTRIBUTION

By definition, the k^th moment about mean is

$$\mu_k = E\{X-E(x)\}^k = \sum_{x=0}^{n} (x-np)^k \, {}^nC_x \, p^x q^{n-x}$$

Differentiating *w.r.t.p*,

$$\frac{d\mu_k}{dp} = -n\,k \sum_{x=0}^{n} {}^nC_x (x-np)^{k-1} p^x q^{n-x} + \sum_{x=0}^{n} {}^nC_x (x-np)^k p^x q^{n-x} \left\{ \frac{x}{p} - \frac{n-x}{q} \right\}$$

$$= -nk \sum_{x=0}^{n} (x-np)^{k-1} p(x) + \frac{1}{pq} \sum_{x=0}^{n} (x-np)^{k+1} p(x)$$

$$\therefore \quad \frac{d\mu_k}{dp} = -nk\,\mu_{k-1} + \frac{1}{pq}\,\mu_{k+1}$$

$$\Rightarrow \quad \mu_{k+1} = pq\left[nk\,\mu_{k-1} + \frac{d\mu_k}{dp} \right].$$

Example 12. *Comment on the following :*

"The Mean of a binomial distribution is 3 and Variance is 4"

Sol. If the given binomial distribution has parameters n and p, then as given

Mean = $np = 3$ ...(1)

& Variance = $npq = 4$ ...(2)

Dividing (2) by (1), we get

$$q = \frac{4}{3}$$

which is impossible, since probability cannot exceed unity. Hence given statement is not correct.

Example 13. *Six dice are thrown 729 times. How many times do you expect at least three dice to show five or six ?*

Sol. p = the chance of getting 5 or 6 with one die = $\frac{2}{6} = \frac{1}{3}$

$$q = 1- \frac{1}{3} = \frac{2}{3}, n = 6, N = 729$$

since dice are in sets of 6 and there are 729 sets.

The binomial distribution is $N(q+p)^n = 729 \left(\frac{2}{3} + \frac{1}{3}\right)^6$

The expected number of times at least three dice showing five or six

$$= 729 \left[{}^6C_3 \left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3 + {}^6C_4 \left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^4 + {}^6C_5 \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^5 + {}^6C_6 \left(\frac{1}{3}\right)^6\right]$$

$$= \frac{729}{3^6} [160+60+12+1] = 233.$$

Example 14. *Out of 800 families with 4 children each, how many families would be expected to have (i) 2 boys and 2 girls (ii) at least one boy (iii) no girl (iv) at most two girls ? Assume equal probabilities for boys and girls.*

Sol. Since probabilities for boys and girls are equal

p = probability of having a boy = $\frac{1}{2}$

q = probability of having a girl = $\frac{1}{2}$

$n = 4$, $N = 800$

The binomial distribution is 800 $\left(\frac{1}{2}+\frac{1}{2}\right)^4$

(i) The expected number of families having 2 boys and 2 girls

$$= 800\ {}^4C_2 \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = 800 \text{ x } 6 \text{ x } \frac{1}{16} = 300.$$

(ii) The expected number of families having at least one boy

$$= 800\left[{}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + {}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= 800 \text{ x } \frac{1}{16}[4 + 6 + 4 + 1] = 750.$$

(iii) The expected number of families having no girl i.e. having 4 boys

$$= 800.\ {}^4C_4\left(\frac{1}{2}\right)^4 = 50.$$

(iv) The expected number of families having at most two girls i.e. having at least 2 boys.

$$= 800\left[{}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 + {}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + {}^4C_4\left(\frac{1}{2}\right)^4\right]$$

$$= 800 \text{ x } \frac{1}{16}[6 + 4 + 1] = 550.$$

10.12 POISSON DISTRIBUTION :

10.12.1 POISSON DISTRIBUTION AS A LIMITING CASE OF BINOMIAL DISTRIBUTION

If the parameters n and p of a binomial distribution are known, we can find the distribution. However, if we assume that as $n \rightarrow \infty$ and $p \rightarrow 0$ such that np always remain finite, say λ, we get the Poisson distribution from the binomial distribution.

Now, for a binomial distribution

$$P(X=r) = {}^nC_r q^{n-r} p^r$$

$$= \frac{n(n-1)(n-2)....(n-r+1)}{r\,!} \text{ x } (1-p)^{n-r} \text{ x } p^r$$

$$= \frac{n(n-1)(n-2)....(n-r+1)}{r\,!} \text{ x } \left(1-\frac{\lambda}{n}\right)^{n-r} \text{ x } \left(\frac{\lambda}{n}\right)^{r}$$

since, $np = \lambda$, $p = \frac{\lambda}{n}$

$$= \frac{\lambda^r}{r\,!} \text{ x } \frac{n(n-1)(n-2)....(n-r+1)}{n^r} \text{ x } \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{r}}$$

$$= \frac{\lambda^r}{r\,!}\left(\frac{n}{n}\right)\left(\frac{n-1}{n}\right)\left(\frac{n-2}{n}\right)......\left(\frac{n-r+1}{n}\right) \text{ x } \frac{\left(1-\frac{\lambda}{n}\right)^{n}}{\left(1-\frac{\lambda}{n}\right)^{r}}$$

$$= \frac{\lambda^r}{r\,!}\left(1-\frac{1}{n}\right),\left(1-\frac{2}{n}\right),......,\left(1-\frac{r-1}{n}\right) \text{ x } \frac{\left[\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda}}{\left(1-\frac{\lambda}{n}\right)^{r}}$$

As $n \to \infty$,each of the (r-1) factors

$\left(1-\frac{1}{n}\right)$, $\left(1-\frac{2}{n}\right)$, ...,$\left(1-\frac{r-1}{n}\right)$ tends to 1. Also $\left(1-\frac{\lambda}{n}\right)^{r}$ tends to 1.

Since $\underset{x \to \infty}{\text{Lt.}} \left(1+\frac{1}{x}\right)^{x} = e$,

$$\therefore \qquad \left[\left(1-\frac{\lambda}{n}\right)^{-n/\lambda}\right]^{-\lambda} \longrightarrow e^{-\lambda} \text{ as } n \to \infty$$

Hence, in the limiting case when $n \to \infty$, we have

$$P(X = r) = \frac{\lambda^r e^{-\lambda}}{r\,!} \qquad (r = 0,1,2,3,......) \qquad .....(1)$$

where λ is a finite number = np.

(1) represents a probability distribution which is called the Poisson probability distribution.

Note 1. λ is called the parameter of the distribution.

Note 2. The sum of the probabilites P(r) for r = 0,1,2,3,...... is 1, since

$$P(0) + P(1) + P(2) + P(3) + ...$$

$$= e^{-\lambda} + \frac{\lambda e^{-\lambda}}{1!} + \frac{\lambda^2 e^{-\lambda}}{2!} + \frac{\lambda^3 e^{-\lambda}}{3!} + ... = e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + ...\right] = e^{-\lambda} . e^{\lambda} = 1.$$

10.12.1 RECURRENCE RELATION FOR THE POISSON DISTRIBUTION

For Poisson distribution, $P(r) = \frac{\lambda^r e^{-\lambda}}{r!}$ and $P(r + 1) = \frac{\lambda^{r+1} e^{-\lambda}}{(r+1)!}$

$$\frac{P(r+1)}{P(r)} = \frac{\lambda r\,!}{(r + 1)!} = \frac{\lambda}{(r + 1)}$$

or $$P(r + 1) = \frac{r}{r + 1}\, P(r),\ r = 0, 1,2,3, ...$$

This is called the recurrence relation for the Poisson distribution.

10.12.2 MEAN AND VARIANCE OF THE POISSON DISTRIBUTION

For the Poisson distribution, $P(r) = \frac{\lambda^r e^{-\lambda}}{r!}$

$$\text{Mean, } \mu = \sum_{r=0}^{\infty} rP(r) = \sum_{r=0}^{\infty} r . \frac{\lambda^r e^{-\lambda}}{r!}$$

$$= e^{-\lambda} \sum_{r=1}^{\infty} \frac{\lambda^r}{r!} = e^{-\lambda}\left[\lambda + \frac{\lambda^2}{1!} + \frac{\lambda^3}{2!} + ...\right]$$

$$= \lambda e^{-\lambda}\left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + ...\right] = \lambda e^{-\lambda} . e^{\lambda} = \lambda$$

Thus, the mean of the Poisson distribution is equal to the parameter λ .

$$\text{Variance, } \sigma^2 = \sum_{r=0}^{\infty} r^2 P(r) - \mu^2 = \sum_{r=0}^{\infty} r^2 . \frac{\lambda^r e^{-\lambda}}{r!} - \lambda^2 = e^{-\lambda} \sum_{r=1}^{\infty} \frac{r^2 \lambda^r}{r!} - \lambda^2$$

$$= e^{-\lambda}\left[\frac{1^2 . \lambda}{1!} + \frac{2^2 . \lambda^2}{2!} + \frac{3^2 . \lambda^3}{3!} + \frac{4^2 . \lambda^4}{4!} + ...\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1 + \frac{2\lambda}{1!} + \frac{3\lambda^2}{2!} + \frac{4\lambda^3}{3!} + ...\right] - \lambda^2$$

$$= \lambda e^{-\lambda}\left[1+\frac{(1+1)\lambda}{1!}+\frac{(1+2)\lambda^2}{2!}+\frac{(1+3)\lambda^3}{3!}+......\right]-\lambda^2$$

$$= \lambda e^{-\lambda}\left[\left(1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+........\right)+\left(\frac{\lambda}{1!}+\frac{2\lambda^2}{2!}+\frac{3\lambda^3}{3!}+.....\right)\right]-\lambda^2$$

$$= \lambda e^{-\lambda}\left[e^{\lambda}+\lambda\left(1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+......\right)\right]-\lambda^2$$

$$= \lambda e^{-\lambda}\left[e^{\lambda}+\lambda e^{\lambda}\right]-\lambda^2 = \lambda e^{-\lambda}\,.\,e^{\lambda}(1+\lambda)-\lambda^2 = \lambda\,.$$

Hence, the variance of the Poisson distribution is also λ .

Thus, the mean and the variance of the Poisson distribution are each equal to the parameter λ .

Note. The mean and the variance of the Poisson distribution can also be derived from those of the binomial distribution in the limiting case when $n \to \infty$, $p \to 0$ and $np = \lambda$.

Mean of binomial distribution is *np*.

$\therefore$ Mean of Poisson distribution $= \underset{n\to\infty}{\text{Lt.}}\, np = \underset{n\to\infty}{\text{Lt.}}\, \lambda = \lambda$

Variance of binomial distribution is $npq = np(1-p)$

$\therefore$ Variance of Poisson distribution $= \underset{n\to\infty}{\text{Lt.}}\, np(1-p) = \underset{n\to\infty}{\text{Lt.}}\, \lambda\left(1-\frac{\lambda}{n}\right) = \lambda$.

Example 15. *If the variance of the Poisson distribution is 2, find the probabilities for r = 1,2,3,4 from the recurrence relation of the Poisson distribution.*

Sol. λ the parameter of Poisson distribution = Variance = 2

Recurrence relation for the Poisson distribution is

$$P(r+1) = \frac{\lambda}{r+1} P(r) = \frac{2}{r+1} P(r) \qquad ...(1)$$

Now $P(r) = \dfrac{\lambda^r e^{-\lambda}}{r!}$ L $P(0) = \dfrac{e^{-2}}{0\,!} = e^{-2} = 0.1353$

Putting r = 0,1,2,3 in (1), we get

$$P(1) = 2P(0) = 2 \times 0.1353 = 0.2706$$

$$P(2) = \frac{2}{2} P(1) = 0.2706$$

$$P(3) = \frac{2}{3} P(2) = \frac{2}{3} \times 0.2706 = 0.1804$$

$$P(4) = \frac{2}{4} P(3) = \frac{1}{4} \times 0.1804 = 0.0902.$$

Example 16. *Assume that the probability of an individual coalminer being killed in a mine accident during a year is* $\frac{1}{2400}$ *. Use Poisson's distribution to calculate the probability that in a mine employing 200 miners there will be at least one fatal accident in a year.*

Sol. Here $p = \frac{1}{2400}$, $n = 200$

$$\therefore \quad \lambda = np = \frac{200}{2400} = \frac{1}{12} = .083$$

$$\therefore \quad P(r) = \frac{\lambda^r e^{-\lambda}}{r!} = \frac{(.083)^r e^{-0.83}}{r!}$$

P(at least one fatal accident) = 1-p(no fatal accident)

$$= 1 - P(0) = 1 - \frac{(.083)^0 e^{-0.83}}{0\,!} = 1 - .92 = 0.08.$$

Example 17. *Six coins are tossed 6400 times. Using the Poisson distribution, determine the approximate probability of getting six heads x times.*

Sol. Probability of getting one head with one coin = $\frac{1}{2}$

$\therefore$ The probability of getting six heads with six coins = $\left(\frac{1}{2}\right)^6 = \frac{1}{64}$

$\therefore$ Average number of six heads with six coins in 6400 throws = $np = 6400 \times \frac{1}{64}$ = 100

$\therefore$ The mean of the Poisson distribution = 100.

Approximate probability of getting six heads *x* times when the distribution is Poisson

$$= \frac{m^x e^{-m}}{x!} = \frac{(100)^x e^{-100}}{(100)!}$$

10.13 NORMAL DISTRIBUTION :

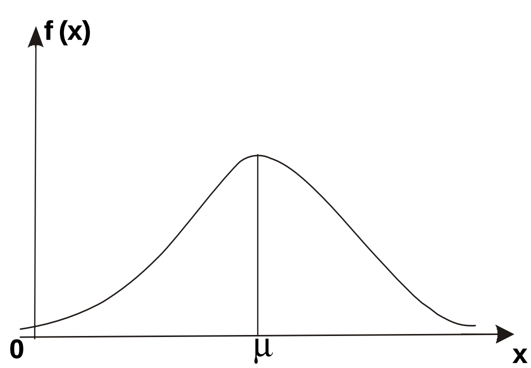
The normal distribution is a continuous distribution. It can be derived from the binomial distribution in the limiting case when *n*, the number of trials is very large and *p*, the probability of a success, is close to $\frac{1}{2}$. The general equation of the normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

where the variable *x* can assume all values from $-\infty$ to $+\infty$, μ and σ, called the parameters of the distribution, are respectively the mean and the standard deviation of the distribution and $-\infty < \mu < \infty$, $\sigma > 0$, *x* is called the normal variate and $f(x)$ is called probability density function of the normal distribution.

If a variable *x* has the normal distribution with mean μ and standard deviation σ, we briefly write $x : N(\mu, \sigma^2)$.

The graph of the normal distribution is called the normal curve. It is bell-shaped and symmetrical about the mean μ .



The two tails of the curve extend to + ∞ and -∞ towards the positive and negative directions of the x-axis, respectively and gradually approach the x-axis without ever metting it. The curve is unimodal and the mode of the normal distribution coincides with its mean μ . The line $x = \mu$ divides the area under the normal curve above x-axis into two equal parts. Thus, the median of the distribution also coincides with its mean and mode. The area under the normal curve between any two given ordinates $x = x_1$ and $x = x_2$ represents the probability of values falling into the given interval. The total area under the normal curve above the x-axis is 1.

10.13.1 BASIC PROPERTIES OF THE NORMAL DISTRIBUTION

The probability density function of the normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

(i) $f(x) \geq 0$ $\int_{-\infty}^{\infty} f(x)\, dx = 1,$

i.e., the total area under the normal curve above the x-axis is 1.

(iii) The curve is bell-shaped. The normal distribution is symmetrical about its mean, *i.e.* about $x = \mu$

(iv) The mean, mode and median of this distribution coincide.

(v) Since $f(x)$ being probability, can never be negative, no portion of curve lies below x-axis.

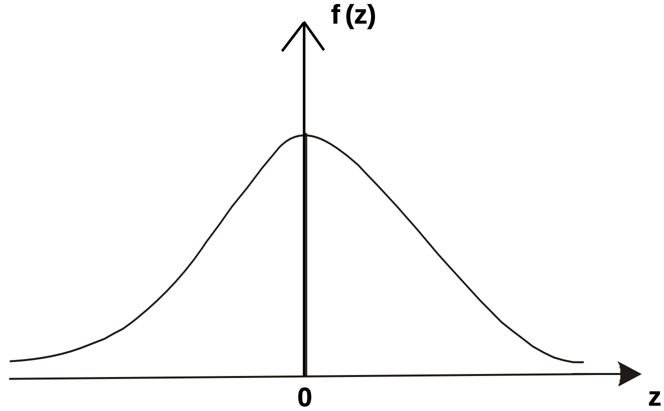
(vi) Area Property : $P(\mu - \sigma < x < \mu + \sigma) = 0.6826$

$$P(\mu - 2\sigma < x < \mu + 2\sigma) = 0.9544$$

$$P(\mu - 3\sigma < x < \mu + 3\sigma) = 0.9973$$

10.13.2 STANDARD FORM OF THE NORMAL DISTRIBUTION

If X is a normal random variable with mean μ and standard deviation σ, then the random variable $Z = \dfrac{X - \mu}{\sigma}$ has the normal distribution with mean 0 and standard deviation 1.



The random variable Z is called the *standard* normal variate.

The probability density function for the normal distribution in standard form is given by

$$f(z) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}, \qquad (-\infty < z < \infty)$$

It is free from any parameter. This helps us to compute areas under the normal probability curve by making use of standard tables.

Note 1. If $f(z)$ is the probability density function for the normal distribution, then

$$P(z_1 \le Z \le z_2) = \int_{z_1}^{z_2} f(z)\, dz = F(z_2) - F(z_1), \text{ where } F(z) = \int_{-\infty}^{z} f(z)\, dz = P(Z \le z)$$

The function F(z) defined above is called the *distribution function* for the normal distribution.

Note 2. The probabilities $P(z_1 \le Z \le z_2)$, $P(z_1 < Z \le z_2)$, $P(z_1 \le Z < z_2)$ and $P(z_1 < Z < z_2)$ are all regarded to be same.

Note 3. $F(-z_1) = 1 - F(z_1)$

Example 18. *A sample of 100 dry battery cells tested to find the length of life produced the following results:*

$$\bar{x} = 12 \text{ hours}, \quad \sigma = 3 \text{ hours}$$

Assuming the data to be normally distributed, what percentage of battery cells are expected to have life.

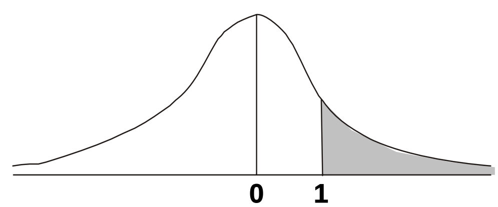
(i) more that 15 hours *(ii) less than 6 hours*

(iii) between 10 and 14 hours ?

Sol. Here x denotes the length of life of dry battery cells.

Also $$z = \frac{x - x}{\sigma} = \frac{x - 12}{3}$$

(i) When $x = 15$, $z = 1$

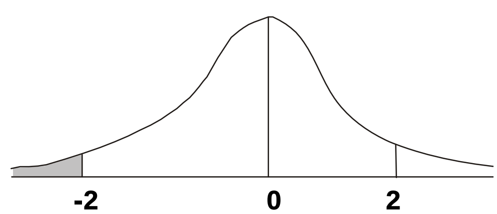


$\therefore$ $P(x > 15) = P(z > 1)$

$= P(0 < z < \text{T}) - P(0 < z < 1)$

$= .5 - 0.3413 = 0.1587 = 15.87\%$.

(ii) When $x = 6$, $z = -2$



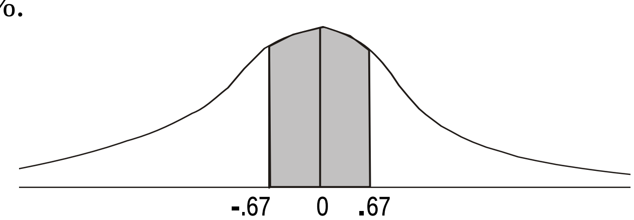
$\therefore$ $P(x < 6) = P(z < -2)$

$= P(z > 2) = P(0 < z < \text{T}) - P(0 < z < 2)$

$= .5 - 0.4772 = 0.0228 = 2.28\%$.

(iii) When $x = 10$, $z = -\frac{2}{3} = -0.67$

When $x = 14$, $z = \frac{2}{3} = 0.67$



$P(10 < x < 14)$

$= P(-0.67 < z < 0.67)$

$= 2P(0 < z < 0.67) = 2 \times 0.2487 = 0.4974 = 49.74\%$.

Example 19. X is normal Variate with mean 30 and S.D. 5 Find the probability that

(i) 26 7 x , ≤ 40, (ii) $X \geq 45$ (iii) $|X-30| > 5$

Sol. Here $\mu = 30$ and $\sigma = 5$

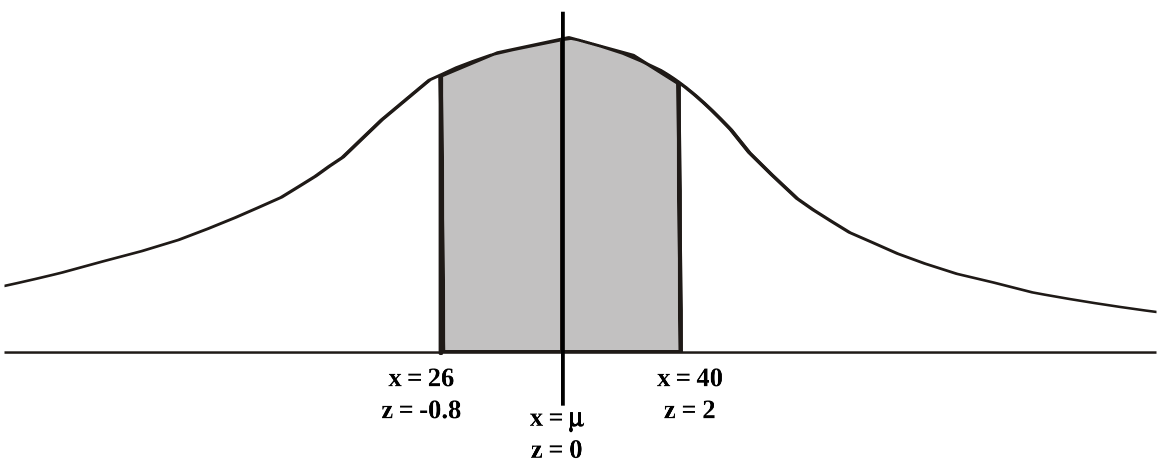
(i) When X = 26, $Z = \dfrac{X - \mu}{\sigma} = \dfrac{26 - 30}{5} = -0.8$

and when X = 40, $Z = \dfrac{40 - 30}{5} = 2$

$P(26 \leq X \leq 40) = P(-0.8 \leq Z \leq 2) = P(-0.8 \leq Z \leq 0) + P(0 \leq Z \leq 2)$

∴ $= P(0 \leq Z \leq 0.8) + 0.47772$

$= 0.2881 + 0.4772 = 0.7653$



(ii) when X = 45, $Z = \dfrac{45 - 30}{2} = 3$

$P(X \geq 45) = P(Z \geq 3) = 0.5 - P(0 \leq Z \leq 3) = 0.5 - 0.49865 = 0.00135$.

(iii) $P(|X-30| \leq 5) = P(25 \leq X \leq 35) = P(-1 \leq Z \leq 1)$

$= 2P(0 \leq Z \leq 1) = 2 \times 0.3413 = 0.6826$

$P(|X-30| > 5) = 1 - P(|X-30| \leq 5)$

$= 1 - 0.6826$

$= 0.3174$

Example 20. *In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and standard deviation of the distribution.*

Sol. Let $\bar{x}$ and be the mean and S.D. respectively.

31% of the items are under 45.

$\Rightarrow$ Area to the left of the ordinate $x = 45$ is 0.31

When $x = 45$, let $z = z_1$

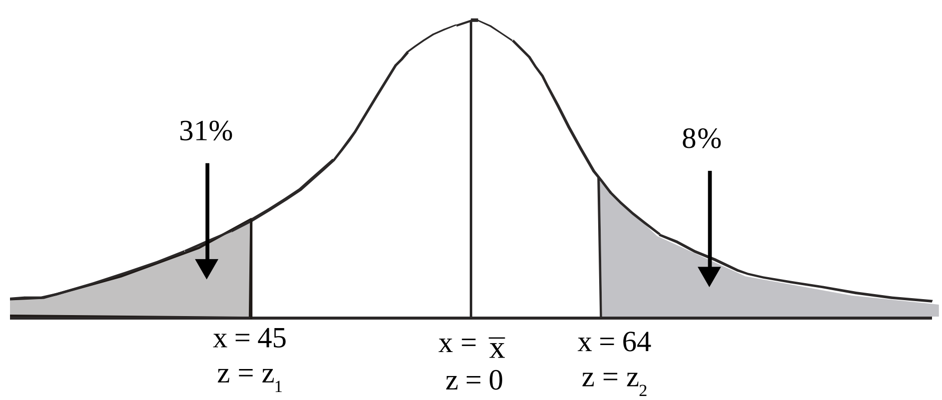
$P(z_1 < z < 0) = .5 - .31 = .19$

From the tables, the value of z corresponding to this area is 0.5

$z_1 = -0.5 [z_1 < 0]$

When $x = 64$, let $z = z_2$

$P(0 < z < z_2) = .5 - .08 = .42$



From the tables, the value of *z* corresponding to this area is 1.4.

$$z_2 = 1.4$$

Since $z = \dfrac{x - \bar{x}}{\sigma}$

$$-0.5 = \frac{45 - \bar{x}}{\sigma} \quad \text{and} \quad 1.4 = \frac{64 -}{\sigma}$$

$\Rightarrow$ 45 - = -0.5σ ...(1)

and 64 - = 1.4σ ...(2)

Subtracting -19 = -1.9σ ∴ σ = 10

From (1) , 45 - = -0.5 x 10 = -5 ∴ = 50.

10.14 SELF ASSESSMENT QUESTIONS :

1. The probability that a contractor will get a plumbing contract is 2/3 and the probability that he will not get an electric contract is 5/9. If the probability of getting atleast one contract is 4/5, what is the probability that he will get both ?

2. A study showed that 65 percent of managers had some business education and 50 percent had some engineering education. Furthermore, 20 percent of the managers had some business education but no engineering education. What is the probability that a manager had some business education, given that he had some engineering education ?

3. A problem in statistics is given to five students, A,B,C,D and E. Their chances of solving it are 1/2, 1/3, 1/4, and 1/5 and 1/6, respectively. What is the probability that the problem will be solved?

4. In a certain college 25% of boys and 10% of girls are studying mathematics. The girls constitute 60% of the student body. What is the

probability that mathematics is being studies? If a student is selected at random and is found to be studying mathematics, find the probability that the student is a girl?

5. Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys (ii) 5 girls (iii) either 2 or 3 boys. Asume equal probability for boys and girls.

6. A man hits a farget with probability 1/4.

   (i) Defermine the probability of hitting at least twice when he fires 7 times.

   (ii) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3?

7. A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the probability that a perticular packet will violate the guarantee.

8. Find the area A under the normal curve :

   (i) to the left of z = -1.78

   (ii) to the left of z = 0.56

   (iii) to the right of z = 1.45

   (iv) corresponding to z Ÿ 2.16

   (v) corresponding to -0.80 Ÿ z Ÿ 1.53

   (vi) to the left of z = - 2.52 & to the right of z = 1.83

9. Defermine the expected no. of boys whose weight is

   (i) between 65 & 70 kg

   (ii) grater than or equal to 72 kg

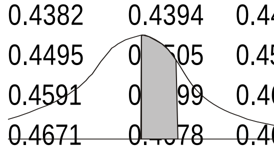
   if the weight X of 800 boys follows normal distribution with $\overline{X}$ = 66, & σ =5

1

Table : Area Under the Normal curve from 0 to z

0 z

z	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0.0040	0.0080	0.120	0.160	0.199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0754
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2258	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2518	0.2549
0.7	0.2580	0.2612	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2996	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993



10. Find the mean & standard deviation of a normal distribution in which 7% of the items are under 35 and 89% of are under 63.

10.15 KEY WORDS :

Probability, Baye's theorem, Probability distributions, Binomial, Poisson, Normal distributions.

10.16 SUGGESTED READINGS :

1. Gupta, S.C. and Kapoor V.K. - Fundamentals of Mathematical Statistics.
2. Grewal, B.S. - Engineering Mathematics
3. Srivastava, K.N. & Dhawan, G.K.- A text book of Engineering Mathematics.
4. Ramana, B.V. - Higher Engineering Mathematics.

E E E

Subject	**:**	**Mathematics-I**	**Paper Code :**	**MCA 103**
Author	**:**	**Prof. Kuldip Bansal**	**Lesson No.**	**: 11**
Lesson	**:**	**Correlation and Regression**		

11.0 OBJECTIVES :

After studing this lesson, you should be able to understand:

* Various methods to study correlation
* Linear regression and lines of regression.
* Properties of regression coefficients
* Curve fitting using method of least squares.

11.1 INTRODUCTION :

So far we have been dealing with distributions which involved only one variate, such distributions were called **univariate distributions.** However, we may have a data where two *characterstics* of each individual will be considered for a group of individuals. Such a data is called a **Bivariate Data.** For example, we may collect data indicating *heights* and *weight* of a group of individuals or ages of husband and wife at the time of their marriage. Given a bivariate data, we shall study that if there exists any type of relation between the two characteristics (variables), i.e., we wish to know how the two variables vary.

11.2 DEFINITIONS :

11.2.1 Bivariate Frequency Distribution

If x_i and y_i denote two variables (i = 1, 2, 3,....n) and the pair (x_i, y_i) occurs f_i times, then f_i is called the **frequency** of the pair (x_i, y_i) and the data showing pairs (x_i, y_i), (i = 1, 2, 3, ...n) together with the frequency f_i is called a **Bivariate Frequency Distribution.**

11.2.2 Correlation

In a bivariate distribution if the two variables vary in such a way that the changes in one are followed by changes in the other, then the variables are said to be correlated. The statistical tool with the help of which the relationship between two or more than two variables is discovered and measured is called **correlation.**

11.2.3 Positive and Negative Correlation

If the increase (decrease) in one variable is followed by the increase(decrease) in the other variable, the correlation is said to be **Positive or Direct.** However, if the increase (decrease) in one variable causes decrease (increase) in the other variable, the correlation is said to be **Negative** or **Inverse.**

11.2.4 Linear Correlation

In the scatter diagram of the points (x_i, y_i) if the points make a straight line trend, the correlation is said to be **Linear.** However, if the points (x_i, y_i) of the scatter diagram do not lie along any straight line and lie along some curve, the correlation is said to be **Non Linear Correlation** or **Curvilinear Correlation.**

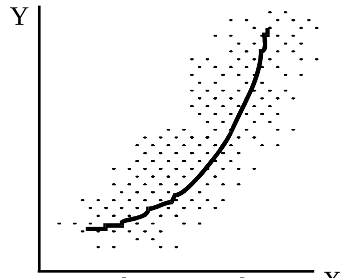
11.3 VARIOUS METHODS TO STUDY CORRELATION :

The various methods of studying correlaton that we shall discuss are :

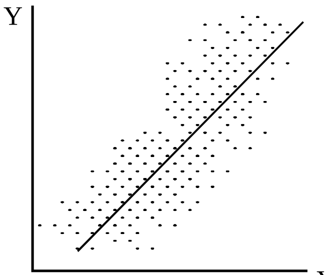
1. The Scatter Points Diagram (Graphic Method).
2. Karl Pearson's Co-efficient of Correlation (Method of finding numerical measure of correlation).
3. The Lines of Regression.

11.3.1 The Scatter Diagram

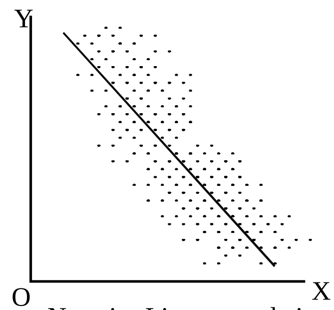
We plot the points (x_i, y_i) (i = 1, 2, 3,....n). The dots of the scatter diagram show the type of correlation between the variables x and y. A few scatter diagrams drawn below show the type of relation between the two variables.



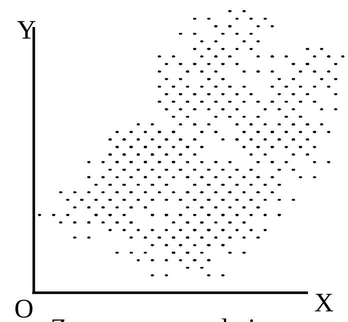
Non-linear correlation



Positive Linear crrelation



Negative Linear correlation



Zero or no correlation

11.3.2 KARL PEARSON'S CO-EFFICIENT OF CORRELATION (OR PRODUCT MOMENT CORRELATION CO-EFFICIENT)

Correlation co-efficient between two variables x and y, usually denoted by r(x, y) or r_{xy} is a numerical measure of linear relationship between them and is defined as

$$r_{xy} = \frac{\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{\Sigma (x_i-\bar{x})^2 \, \Sigma (y_i-\bar{y})^2}} = \frac{\frac{1}{n}\Sigma(x_i-\bar{x})(y_1-\bar{y})}{\sqrt{\frac{1}{n}\Sigma (x_i-\bar{x})^2 . \frac{1}{n}\Sigma (y_i-\bar{y})^2}}$$

$$= \frac{\frac{1}{n}\Sigma(x_i-\bar{x})(y_i-\bar{y})}{\sigma_x\sigma_y} = \frac{COV\,(x, y)}{\sigma_x\sigma_y} \quad , \text{ where}$$

$$\sigma_x = \sqrt{\frac{1}{n}\Sigma (x_i-\bar{x})^2} \quad , \quad \sigma_y = \sqrt{\frac{1}{n}\Sigma (y_i-\bar{y})^2}$$

Note. Correlation co-efficient is independent of change of origin and scale.

Let us define two new variables u and v as

$$u = \frac{x-a}{h}, \quad v = \frac{y-b}{k}$$, where a, b, h, k are constants, then $r_{xy} = r_{uv}$.

11.3.3 COMPUTATION OF CORRELATION CO-EFFICIENT

We know that $$r_{xy} = \frac{\frac{1}{n}\Sigma(x_i-\bar{x})(y_1-\bar{y})}{\sigma_x\sigma_y}$$

Now $$\frac{1}{n}\Sigma(x_i-\bar{x})(y_i-\bar{y}) = \frac{1}{n}\Sigma(x_iy_i-x_i\bar{y}-y_i\bar{x}+\bar{x}\bar{y})$$

$$= \frac{1}{n}\Sigma x_iy_i - \bar{y}.\frac{1}{n}\Sigma x_i - \bar{x}\frac{1}{n}\Sigma y_i + \frac{1}{n}(n\bar{x}\bar{y})$$

$$= \frac{1}{n}\Sigma x_iy_i - \bar{y}.\bar{x} - \bar{x}.\bar{y} + \bar{x}.\bar{y} = \frac{1}{n}\Sigma x_iy_i - \bar{x}.\bar{y}$$

$$\sigma_x^2 = \frac{1}{n}\Sigma(x_i-\bar{x})^2 = \frac{1}{n}\Sigma(x_i^2 - 2x_i\bar{x} + \bar{x}^2) = \frac{1}{n}\Sigma x_i^2 - 2\bar{x}.\frac{1}{n}\Sigma x_i + \frac{1}{n}(n\bar{x}^2)$$

$$= \frac{1}{n}\Sigma x_i^2 - 2\bar{x}.\bar{x} + \bar{x}^2 = \frac{1}{n}\Sigma x_i^2 - \bar{x}^2$$

Similarly, $$\sigma_y^2 = \frac{1}{n}\Sigma y_i^2 - \bar{y}^2$$

$$\therefore \quad r_{xy} = \frac{\frac{1}{n}\Sigma x_iy_i-\bar{x}\bar{y}}{\sqrt{\left(\frac{1}{n}\Sigma x_i^2-\bar{x}^2\right)\left(\frac{1}{n}\Sigma y_i^2-\bar{y}^2\right)}}$$

If $$u = \frac{x-a}{h}, \quad v = \frac{y-b}{k}$$

then $$r_{xy} = r_{uv} = \frac{\frac{1}{n}\Sigma u_iv_i-\bar{u}\bar{v}}{\sqrt{\left(\frac{1}{n}\Sigma u_i^2-\bar{u}^2\right)\left(\frac{1}{n}\Sigma v_i^2-\bar{v}^2\right)}}$$

Note :

(i) $\frac{1}{n}\Sigma (x-\bar{x})(y-\bar{y})$ is known as co-variance of the variable x and y and is denoted by Cov(x, y).

(ii) σ_x and σ_y are, respectively the standard deviations of x and y are essentially positive. Thus, the sign of correlation co-efficient 'r' depends upon the $Cov(x, y) = \frac{1}{n} \Sigma (x-\bar{x})(y-\bar{y})$

Cor. $r = 0$, if and only if $Cov(x,y) = 0$

and then the variables x and y are said to be **uncorrelated** or we say that there is no correlation between the variables x and y.

(iii) Co-efficient of correlation cannot numerically exceed unity,

that is, $|r| \le 1 \Rightarrow -1 \le r \le 1$

If r = 1, the correlation is said to be positive and perfect.

If r = -1, the correlation is said to be negative and perfect.

If r = 0, the variables x and y are said to be **Uncorrected.**

(iv) CALCULATION OF CO-EFFICIENT OF CORRELATION FOR A BIVARIATE FREQUENCY DISTRIBUTION

If the bivariate data on x and y is presented on a two way correlation table and f is the frequency of a particular rectangle in the correlation table, then

$$r_{xy} = \frac{\Sigma fxy - \frac{1}{n}\Sigma fx \, \Sigma fy}{\sqrt{\left(\Sigma fx^2 - \frac{1}{n}(\Sigma fx)^2\right)\left(\Sigma fy^2 - \frac{1}{n}(\Sigma fy)^2\right)}}$$

Since change of origin and scale do not affect the co-efficient of correlation,

$\therefore$ $r_{xy} = r_{uv}$ where the new variables u, v are properly chosen.

Example 1. *Ten students got the following percentage of marks in Papers of Economics and Statistics :*

Roll Nos. :	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Marks in Economics:	*78*	*36*	*98*	*25*	*75*	*82*	*90*	*62*	*65*	*39*
Marks in Statistics :	*84*	*51*	*91*	*60*	*68*	*62*	*86*	*58*	*53*	*47*

Calculate the co-efficient of correlation.

Sol. Let the marks in the two subjects be denoted by x and y respectively.

x	y	u = x-65	v = y-66	u^2	v^2	uv
78	84	13	18	169	324	234
36	51	-29	-15	841	225	435
98	91	33	25	1089	625	825
25	60	-40	-6	1600	36	240
75	68	10	2	100	4	20
82	62	17	-4	289	16	-68
90	86	25	20	625	400	500
62	58	-3	-8	9	64	24
65	53	0	-13	0	169	0
39	47	-26	-19	676	361	494
Total		0	0	5398	2224	2734

$$\bar{u} = \frac{1}{n}\Sigma u_i = 0, \bar{v} = \frac{1}{n}\Sigma v_i = 0$$

$$r_{uv} = \frac{\frac{1}{n}\Sigma u_i v_i - \bar{u}\bar{v}}{\sqrt{\left(\frac{1}{n}\Sigma u_i^2 - \bar{u}^2\right)\left(\frac{1}{n}\Sigma v_i^2 - \bar{v}^2\right)}} = \frac{\frac{1}{10}(2734)}{\sqrt{\frac{1}{10}(5398) \cdot \frac{1}{10}(2224)}}$$

$$ruv = \frac{2734}{\sqrt{5398 \times 2224}} = 0.787$$

Hence $r_{xy} = r_{uv} = 0.787.$

Example 2. *Find the co-efficient of correlation for the following table :*

x :	10	14	18	22	26	30
y :	18	12	24	6	30	36

Sol. Let $u = \frac{x-22}{4}$, $v = \frac{y-24}{6}$.

x	y	u	v	u^2	v^2	uv
10	18	-3	-1	9	1	3
14	12	-2	-2	4	4	4
18	24	-1	0	1	0	0
22	6	0	-3	0	9	0
26	30	1	1	1	1	1
30	36	2	2	4	4	4
Total		-3	-3	19	19	12

$$\bar{u} = \frac{1}{n}\sum u_i = \frac{1}{6}(-3) = -\frac{1}{2}; \quad \bar{v} = \frac{1}{n}\sum v_i = \frac{1}{6}(-3) = -\frac{1}{2}$$

$$r_{uv} = \frac{\frac{1}{n}\sum u_i v_i - \bar{u}\bar{v}}{\sqrt{\left(\frac{1}{n}\sum u_i^2 - \bar{u}^2\right)\left(\frac{1}{n}\sum v_i^2 - \bar{v}^2\right)}}$$

$$= \frac{\frac{1}{6}(12) - \frac{1}{4}}{\sqrt{\left[\frac{1}{6}(19) - \frac{1}{4}\right]\left[\frac{1}{6}(19) - \frac{1}{4}\right]}} = \frac{\frac{7}{4}}{\frac{35}{12}} = \frac{3}{5} = 0.6$$

Hence $r_{xy} = r_{uv} = 0.6.$

Example 3. A computer while calculating correlation co-efficient between two variables X and Y from 25 pairs of observations obtained the following results :

$$n = 25 \qquad \sum X = 125, \qquad \sum X^2 = 650,$$

$$\sum Y = 100, \qquad \sum Y^2 = 460, \qquad \sum XY = 508.$$

It was, however, later discovered at the time of checking that he had copied down two

X	Y
6	14
8	6

pairs as while the correct values were

X	Y
8	12
6	8

Obtain the correct values of correlation co-efficient.

Sol. Corrected $\sum X = 125-6-8+8+6 = 125$

Corrected $\sum Y = 100-14-6+12+8 = 100$

Corrected $\sum X^2 = 650-6^2-8^2+8^2+6^2 = 650$

Corrected $\sum Y^2 = 460-14^2-6^2+12^2+8^2 = 436$

Corrected $\sum XY = 580-6 \times 14-8 \times 6+8\times 12+6\times 8 = 520$

(subtract the incorrect values and add the corresponding correct values)

$$\overline{X} = \frac{1}{n}\sum X = \frac{1}{25} \times 125 = 5$$

$$\overline{Y} = \frac{1}{n}\sum Y = \frac{1}{25} \times 100 = 4$$

Corrected $$r_{xy} = \frac{\frac{1}{n}\sum XY - \overline{X}\,\overline{Y}}{\sqrt{\left(\frac{1}{n}\sum X^2 - \overline{X}^2\right)\left(\frac{1}{n}\sum Y^2 - \overline{Y}^2\right)}}$$

$$r_{xy} = \frac{\frac{1}{25} \times 520 - 5 \times 4}{\sqrt{\left(\frac{1}{25} \times 650 - 25\right)\left(\frac{1}{25} \times 436 - 16\right)}}$$

$$rxy = \frac{\frac{4}{5}}{\sqrt{(1)\left(\frac{36}{25}\right)}} = \frac{4}{5} \times \frac{5}{6} = 0.67.$$

11.4 REGRESSION :

After finding the fact of correlation between two variables, we can know the extent to which one variable varies in response to a given variation in the other variable i.e., we can know the nature of relationship between the two variables.

Regression measures the nature and extent of correlation.

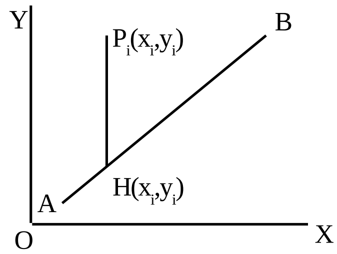
11.4.1 LINEAR REGRESSION

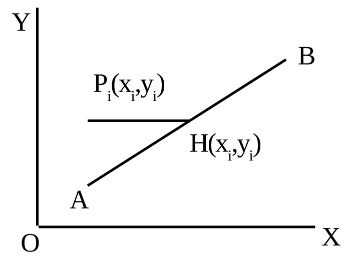
If two variates x and y are correlated i.e., there exists an association or relationship between them, then the scatter diagram will be more or less concentrated round a curve. This curve is called the *curve of regression* and the relationship is said to be expressed by means of curvilinear regression. In the particular case, when the curve is a straight line, it is called a *line of regression* and the regression is said to be linear.

A line of regression is the straight line which gives the best fit in the least square sense to the given frequency.

If the line of regression is so chosen that the sum of squares of deviation parallel to the axis of y is minimised [*see Fig. (a)*] , it is called *the line of regression of y on x* and it gives *the best estimate of y for any given value of x.*

If theline of regression is so chosen that the sum of squares of deviations parallel to the axis of x is minimised [*see Fig. (b)*], it is called *the line of regression of x on y* and it gives the best estimate of x for any given value of y.





11.4.2 LINES OF REGRESSION

Let the equation of line of regression of y on x be

$$y = a+bx \qquad ...(1)$$

Then $$\bar{y} = a+b\bar{x} \qquad ...(2)$$

Substracting (2) from (1), we have

$$y-\bar{y} = b(x-\bar{x}) \qquad ...(3)$$

The normal equations are $\Sigma y = na+b\Sigma x$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \qquad ...(4)$$

Shifting the origin to $(\bar{x}, \bar{y})$, (4) becomes

$$\Sigma(x-\bar{x})(y-\bar{y}) = a\Sigma(x-\bar{x})+b\Sigma(x-\bar{x})^2 \qquad ...(5)$$

Since $$\frac{\Sigma(x_i-\bar{x})(y_1-\bar{y})}{n\sigma_x\sigma_y} = r$$

$$\Sigma(x-\bar{x}) = 0$$

$$\frac{1}{n}\Sigma(x-\bar{x})^2 = \sigma_x^{\ 2}$$

$\therefore$ From (5), $nr\sigma_x\sigma_y = a.0 + b.n\sigma_x^{\ 2} \quad \Rightarrow \quad b = \dfrac{r\sigma_y}{\sigma_x}$

Hence, from (3), the line of regression of y on x is $y-\bar{y} = r\dfrac{\sigma_y}{\sigma_x}(x-\bar{x})$

Similarly, the line of regression of x and y is $x-\bar{x} = r\frac{\sigma_x}{\sigma_y}(y-\bar{y})$

$\frac{r\sigma_y}{\sigma_x}$ is called the regression co-efficient of y on x and is denoted by b_{yx}.

$\frac{r\sigma_x}{\sigma_y}$ is called the regression co-efficient of x on y and is denoted by b_{xy}.

Note. If r = 0, the two lines of regression become $y = \bar{y}$ and $x = \bar{x}$ which are two straight lines parallel to X and Y axes respectively and passing through their means $\bar{y}$ and $\bar{x}$. They are mutually perpendicular.

If $r = \pm 1$, the two lines of regression will coincide.

11.4.3 PROPERTIES OF REGRESSION CO-EFFICIENTS

Property I. Correlation co-efficient is the geometric mean between the regression co-efficients.

Proof. The co-efficients of regression are $\frac{r\sigma_y}{\sigma_x}$ and $\frac{r\sigma_x}{\sigma_y}$.

G.M. between them $= \sqrt{\frac{r\sigma_y}{\sigma_x} \times \frac{r\sigma_x}{\sigma_y}} = \sqrt{r^2} = r$

= co-efficient of correlation.

Property II. If one of the regression co-efficients is greater than unity, the other must be less than unity.

Proof. The two regression co-efficients are $b_{yx} = \frac{r\sigma_y}{\sigma_x}$ and $b_{xy} = \frac{r\sigma_x}{\sigma_y}$.

Let $b_{yx} > 1$, then $\frac{1}{b_{yx}} < 1$, Since $b_{yx} . b_{xy} = r^2 \leq 1$ $[\because -1 \leq r \leq 1]$

$\therefore \quad b_{xy} \leq \frac{1}{b_{yx}} < 1$

Similarly, if $b_{xy} > 1$, then $b_{yx} < 1$.

Property III. Arithmetic mean of regression co-efficients is greater than the correlation co-efficient.

Proof. We have to prove that $\frac{b_{yx}+b_{xy}}{2} > r$ or $\frac{\frac{r\sigma_y}{\sigma_x} + \frac{r\sigma_x}{\sigma_y}}{2} > r$

or $\sigma_y^2 + \sigma_x^2 > 2\sigma_x\sigma_y$ or $(\sigma_x-\sigma_y)^2 > 0$ which is true.

Property IV. Regression co-efficients are independent of the origin but not of scale.

Proof. Let $u = \frac{x-a}{h}$, $v = \frac{y-b}{k}$ where a, b, h and k are constants

$$b_{yx} = \frac{r\sigma_y}{\sigma_x} = r \cdot \frac{k\sigma_v}{h\sigma_u} = \frac{k}{h}\left(\frac{r\sigma_v}{\sigma_u}\right) = \frac{k}{h} b_{vu}$$

Similarly, $b_{xy} = \frac{h}{k} b_{uv}$.

Thus, b_{yx} and b_{xy} are both independent of 'a' and 'b' but not of h and k.

Property V. The correlation co-efficient and the two regression co-efficients have same sign.

Proof. Regression co-efficient of y on x = $b_{yx} = r\frac{\sigma_y}{\sigma_x}$

Regression co-efficient of x on y = $b_{xy} = r\frac{\sigma_x}{\sigma_y}$

Since σ_x and σ_y are both positive, b_{yx}, b_{xy} and r have same sign.

11.4.4 ANGLE BETWEEN TWO LINES OF REGRESSION

If θ is the acute angle between the two regression lines in the case of two variables x and y, show that

$$\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$ where r, σ_x, σ_y have their usual meanings.

Explain the significance of the formula when r = 0 and r = ±1.

Proof. Equations to the lines of regression of y on x and x on y are

$$y-\bar{y} = \frac{r\sigma_y}{\sigma_x}(x-\bar{x}) \text{ and } x-\bar{x} = \frac{r\sigma_x}{\sigma_y}(y-\bar{y})$$

Their slopes are $m_1 = \frac{r\sigma_y}{\sigma_x}$ and $m_2 = \frac{\sigma_y}{r\sigma_x}$

$$\therefore \qquad \tan\theta = \pm \frac{m_2-m_1}{1+m_2m_1} = \pm \frac{\frac{\sigma_y}{r\sigma_x} - \frac{r\sigma_y}{\sigma_x}}{1 + \frac{\sigma_y^2}{\sigma_x^2}}$$

$$= \pm \frac{1-r^2}{r} \cdot \frac{\sigma_y}{\sigma_x} \cdot \frac{\sigma_x^2}{\sigma_x^2+\sigma_y^2} = \pm \frac{1-r^2}{r} \cdot \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

Since $r^2 \leq 1$ and σ_x, σ_y are positive. Therefore, positive sign gives the acute angle between the lines. Hence $\tan\theta = \frac{1-r^2}{r} \cdot \frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$

(i) When r = 0, θ = $\frac{\pi}{2}$

∴ The two lines of regression are perpendicular to each other.

Hence the estimated value of y is the same for all values of x and *vice versa*.

(ii) When = ± 1, tan θ = 0 so that, θ = 0 or π.

Hence the lines of regression coincide and there is perfect correlation between the two variates x and y.

Note.

$$\frac{r\sigma_x}{\sigma_y} = \frac{\frac{1}{n}\sum xy - \bar{x}\bar{y}}{\sigma_y^2} = \frac{\frac{1}{n}\sum xy - \bar{x}\bar{y}}{\frac{1}{n}\sum y^2 - \bar{y}^2}$$

Similarly, the value of $\frac{r\sigma_y}{\sigma_x}$ can be written.

Example 4. *Calculate the co-efficient of correlation and obtain the least square regression line of y on x for the following data* :

x	:	1	2	3	4	5	6	7	8	9
y	:	9	8	10	12	11	13	14	16	15

Also obtain an estimate of y which should correspond on the average to x = 6.2.

x	y	u = x-5	v = y-12	u^2	v^2	uv
1	9	-4	-3	16	9	12
2	8	-3	-4	9	16	12
3	10	-2	-2	4	4	4
4	12	-1	0	1	0	0
5	11	0	-1	0	1	0
6	13	1	1	1	1	1
7	14	2	2	4	4	4
8	16	3	4	9	16	12
9	15	4	3	16	9	12
Total		0	0	60	60	57

$$r_{xy} = r_{uv} = \frac{\frac{1}{n}\sum uv - \bar{u}\bar{v}}{\sqrt{\left(\frac{1}{n}\sum u^2 - \bar{u}^2\right)\left(\frac{1}{n}\sum v^2 - \bar{v}^2\right)}}$$

$$= \frac{\frac{1}{9}(57) - 0}{\sqrt{\left(\frac{1}{9}(60) - 0\right)\left(\frac{1}{9}(60) - 0\right)}}$$

$$= \frac{\frac{1}{9}(57)}{\frac{1}{9}(60)} = \frac{19}{20} = 0.95 \text{ , and } \frac{r\sigma_y}{\sigma_x} = \frac{r\sigma_v}{\sigma_u} = 0.95$$

Also $\bar{x} = 5 + \frac{1}{9}\Sigma u = 5, \bar{y} = 12 + \frac{1}{9}\Sigma v = 12$

Equation of the line of regression of y on x is

$$y-\bar{y} = \frac{r\sigma_y}{\sigma_x}(x-\bar{x}) \text{ or } y-12 = 0.95\,(x-5) \text{ or } y = 0.95x+7.25$$

When x = 6.2, estimated value of y = 0.95 × 6.2 × 7.25 = 5.89 + 7.25 = 13.14.

Example 5. *In a partialy destroyed laboratory record of an analysis of a correlation data, the following results only are legible :*

Variance of x = 9

Regression equations : 8x-10y+66 = 0, 40x-18y = 214.

What were (a) the mean values of x and y, (b) the standard deviation of y, and (c) the co-efficient of correlation between x and y.

Sol. (i) **Since both the lines of regression pass through the point** $(\bar{x}, \bar{y})$ therefore, we have

$$8\bar{x}-10\bar{y}+66 = 0 \qquad ...(1)$$

$$40\bar{x}-18\bar{y}-214 = 0 \qquad ...(2)$$

Multiplying (1) by 5, $40\bar{x}-50\bar{y}+330 = 0$ ...(3)

Substracting (3) from (2), $32\bar{y}-544 = 0$

$$\therefore \quad y = \frac{544}{32} = 17$$

∴ From (1), $8\bar{x}-170+66 = 0$ or $8\bar{x} = 104$ ∴ $\bar{x} = 13$

Hence $\bar{x} = 13, \bar{y} = 17$ Which gives (a)

(ii) Variance x = $\sigma_x^2 = 9$ (given)

∴ $\sigma_x = 3$

The equations of lines of regression can be written as

$$y = 0.8x+6.6 \quad \text{and} \quad x = 0.45y+5.35$$

∴ The regression co-efficient of y on x is $\frac{r\sigma_y}{\sigma_x} = .8$ ...(4)

The regression co-efficient of x on y is $\frac{r\sigma_x}{\sigma_y} = 0.45$ ...(5)

Multiplying (4) and (5), $r^2 = .8 \times .45 = .36$

$$r = 0.6, \text{ which gives (b)}$$

(+ve sign with sq. root is taken because regression co-efficients are +ve).

From (4), $\sigma_y = \frac{.8\sigma_x}{r} = \frac{.8 \times 3}{0.6} = 4.$, which gives (c)

11.5 CURVE FITTING :

Suppose we are given a data in terms of two variables x and y. The problem of finding an analytic expression of the form y = f(x) which fits the given data is called curve fitting.

11.5.1 BEST FITTING CURVE

Let the given data be represented by a set of n points (x_i, y_i), i = 1, 2, 3, ...,n. Let y = f(x) be an approximate curve which fits the given set of data.

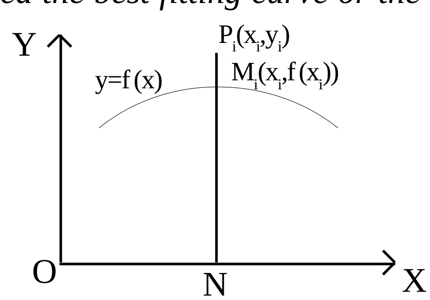
Let $Y_i = f(x_i)$ then Y_i is called the **expected value** of y corresponding to $x = x_i$. The value y_i is called the **observed value** of y corresponding to $x = x_i$.

In general $Y_i \neq y_i$, as the point $P_i(x_i, y_i)$ does not necessarily lie on the curve y = f(x), $E_i = y_i - Y_i$ *is called the error of estimate or the residual for* y_i.

of all curves approximating a given set of points, the curve for which

$$E = E_1^2 + E_2^2 + ... + E_n^2 = \sum_{i=1}^{n} E_i^2$$

is a minimum is called the best fitting curve or the least square curve.



11.5.2 METHOD OF LEAST SQUARES

Let (x_i, y_i), i = 1, 2, 3, ..., n be a set of n points and let

$$y = a_0+a_1x+a_2x^2 \qquad ...(1)$$

be the parabola of best fit to the set of n given points.

Observed value of y corresponding to $x = x_i$ is y_i.

Expected value of y corresponding to $x = x_i$ is given by

$$Y_i = a_0+a_1x_i+a_2x_i^2$$

Error of estimate for y_i is given by

$$E_i = y_i-Y_i = y_i-a_0-a_1x_i-a_2x_i^2$$

Since (1) is the polynomial of best fit

$$\therefore \qquad E = E_1^2+E_2^2+......+E_n^2 = \sum_{i=1}^{n} E_i^2$$

$$= \sum_{i=1}^{n} (y_i-a_0-a_1x_i-a_2x_i^2) \text{ is minimum.}$$

From the principle of maxima and minima, the partial derivatives of E w.r.t. a_0, a_1, a_2 should vanish separately.

i.e.
$$\frac{\partial E}{\partial a_0} = \frac{\partial E}{\partial a_1} = \frac{\partial E}{\partial a_2} = 0$$

$$\frac{\partial E}{\partial a_0} = -2\,\Sigma(y_i-a_0-a_1x_i-a_2x_i^2) = 0$$

$$\frac{\partial E}{\partial a_1} = -2\,\Sigma x_i\,(y_i-a_0-a_1x_i-a_2x_i^2) = 0$$

$$\frac{\partial E}{\partial a_2} = -2\,\Sigma x_i^2(y_i-a_0-a_1x_i-a_2x_i^2) = 0$$

which on simplification give $\Sigma y_i = na_0+a_1\Sigma x_i+a_2\Sigma x_i^2$

$$\Sigma x_i y_i = a_0\Sigma x_i + a_1\Sigma x_i^2 + a_2\Sigma x_i^3$$

$$\Sigma x_i^2 y_i = a_0\Sigma x_i^2 + a_1\Sigma x_i^3 + a_2\Sigma x_i^4$$

summation extending over i from 1 to n.

Thus we have 3 equations in 3 unknowns a_0, a_1, a_2. Solving these equations simultaneously, we get the values of the constants a_0, a_1, a_2.

These equations are called *normal equations.*

Note: Fitting of a straight line. When the line to be fitted is $y = a_0 + a_1 x$.

The normal equations are
$$\left.\begin{aligned} \Sigma y_i &= na_0 + a_1\Sigma x_i \\ \Sigma x_i y_i &= a_0\Sigma x_i + a_1\Sigma x_i^2 \end{aligned}\right\}$$

Example 6. *Fit a straight line to the following data* :

x :	0	1	2	3	4
y :	1	1.8	3.3	4.5	6.3

Sol. Let the straight line to be fitted to the data be

$$y = a + bx$$

then the normal equations are
$$\Sigma y = na + b\Sigma x,$$
$$\Sigma xy = a\Sigma x + b\Sigma x^2.$$

x	y	xy	x^2
0	1	0	0
1	1.8	1.8	1
2	3.3	6.6	4
3	4.5	13.5	9
4	6.3	25.2	16
10	16.9	47.1	30

Here $n = 5$

On substituting, the normal equations become

$$16.9 = 5a+10b \quad ...(i)$$

$$47.1 = 10a+30b \quad ..(ii)$$

Solving (i) and (ii), we get $a = 0.72$, $b = 1.33$.

Hence the equation of the line of best fit is $y = 0.72+1.33x$.

Example 7. *Fit a straight line to the following data, taking y as the dependent variable:*

x :	1	2	3	4	5	6	7	8	9
y :	9	8	10	12	11	13	14	16	15

Sol. Mean of x-series = 5, Mean of y-series = 12

Let $X = x-5$ and $Y = y-12$

Let the line of best fit be $Y = a+bX$

then normal equations are $\sum Y = na + b\sum X$

$$\sum XY = a\sum X + b\sum X^2$$

x	y	X=x-5	Y=y-12	XY	X^2
1	9	-4	-3	12	16
2	8	-3	-4	12	9
3	10	-2	-2	4	4
4	12	-1	0	0	1
5	11	0	-1	0	0
6	13	1	1	1	1
7	14	2	2	4	4
8	16	3	4	12	9
9	15	4	3	12	16
		0	0	57	60

Here n=9

∴ The normal equations become

$0=9a+0$ i.e. $a=0$

$57 = 0+60b$ i.e. $b=.95$

and

The equation of the line of best fit is

$Y = 0+.95X$ or $y-12 = .95(x-5)$ or $y = 7.25+.95x$.

Example 8. Fit a second degree parabola to the following data :

x :	1	2	3	4	5	6	7	8	9
y :	2	6	7	8	10	11	11	10	9

Sol. Let $X = x-5$, $Y = y-7$

and let the curve of best fit be $Y = a+bX+cX^2$

The normalequations are $\sum Y = 9a + b\sum X + c\sum X^2$

$\sum XY = a\sum X + b\sum X^2 + c\sum X^3$

$\sum X^2Y = a\sum X^2 + b\sum X^3 + c\sum X^4$

x	y	X	Y	XY	X^2	X^3	X^4
1	2	-4	-5	20	16	-64	256
2	6	-3	-1	3	9	-27	81
3	7	-2	0	0	4	-8	16
4	8	-1	1	-1	1	-1	1
5	10	0	3	0	0	0	0
6	11	1	4	4	1	1	1
7	11	2	4	8	4	8	16
8	10	3	3	9	9	27	81
9	9	4	2	8	16	64	256
		0	11	51	60	0	708

The normalequations become $11 = 9a+6c$

$51=60b$

$-9 = 60a+708$

Solving these equations, a=3, b=0.85, c=-0.27

Hence the curve of best fit is $Y=3+0.85X=0.27X^2$

or $y-7=3+0.85(x-5)-0.27(x-5)^2$

$= 3+0.85x-4.25-0.27x^2+2.7x-6.75$

or $y =-1 + 3.55x - 0.27x^2$.

11.6 SELF ASSESSMENT QUESTIONS :

1. Determine r for the following data :

X :	50	60	70	90	100
Y :	65	51	40	26	8

2. In a paired data for x, y with n =25,

$\sum x = 127, \sum y = 100, \sum x^2 =760, \sum y^2 =449, \sum xy = 500,$

it was found later that two pairs of correct values

x	y
8	12
6	8

were (erranaously) copied down as

x	y
8	14
8	6

Determine the correlation coefficient for the corrected data.

3. Determine the regression line of

(i) y on x and (ii) x on y, (iii) Find r using the regression coefficients.

4. For the following data determine

(i) least square regression line of y on x

(ii) least square regression line of x on y

x :	6	5	8	8	7	6	10	4	9	7
y :	8	7	7	10	5	8	10	6	8	6

5. Fit a least squares straight line for the following data :

x :	1	2	3	4	5	6
y :	6	4	3	5	4	2

6. Fit a least square quadratic curve to the following data :

x :	1	2	3	4
y :	1.7	1.8	2.3	3.2

7. Fit a least square stright line to the following data :

x :	2	7	9	1	5	12
y :	13	21	23	14	15	21

11.7 KEY WORDS :

Correlation, regression, method of least squares, Regression Coefficient Scatter Diagram.

11.8 SUGGESTED READINGS :

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