Roll No. ....

Subject Code—3226

M. Sc. EXAMINATION

(First Semester)

MATHEMATICS

MAL-512

Real Analysis

Time : 3 Hours *Maximum Marks* : 100

Note : Attempt any *Five* questions. All questions carry equal marks.

1. (a) Discuss the uniform convergence of the sequence $\{f_n\}$ where $f_n(x) = x^n$, $0 \le x \le 1$.

(b) State and prove Cauchy's criterion for uniform convergence of sequences.

(c) Show that the series $\sum \frac{x}{n^p + x^2 n^q}$ converges uniformally over any finite interval $[a, b]$ for (*i*) $p > 1$, $q \geq 0$. (*ii*) $0 < p \leq 1$, $p + q > 2$.

2. (a) State and prove Dirichlet's tests for uniform convergence of series.

(b) Show that the series $\sum \frac{\cos n\theta}{n^p}$ converges uniformally for all values of $p > 0$ in the interval $[\alpha, 2\pi - \alpha]$ where $0 < \alpha < \pi$.

(c) Let $\sum_0^\infty a_n x^n$ be a power series with unit radius of convergence and let

$$f(x) = \sum_0^\infty a_n x^n, \ -1 < x < 1.$$ If the series $\sum a_n$ converges; then :

$$\operatorname*{Lt}_{x \to 1-0} f(x) = \sum_0^\infty a_n$$

3. (a) State and prove Weierstrass Approximation theorem. 13

(b) Let E be an open set in R^n and f maps E in R^m and $x \in E$. Let $h \in R^n$ is small enough such that $x + h \in E$. Then f has a unique derivative. 7

4. (a) If (*i*) f_x and f_y exist in the neighbourhood of the point (a, b) and (*ii*) f_x and f_y are differentiable at (a, b), then $f_{xy} = f_{yx}$.

(b) If $f(x, y) = \sqrt{|xy|}$, prove that Taylor's expansion about the point (x, x) is not valid in any domain which includes the origin.

5. (a) Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the condition :

$$\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1 \text{ and } z = x + y.$$

(b) If J is the Jacobian of the system u, v with respect to x and y and J′ is the Jocobian of x and y w.r.t. u and v, then show that $JJ' = 1$.

6. (a) If $f_1, f_2 \in R(\alpha)$ and C is a constant, then show that :

(*i*) $(f_1 + f_2) \in R(\alpha)$ and

$$\int_a^b (f_1 + f_2)\, d\alpha = \int_a^b f_1\, d\alpha + \int_a^b f_2\, d\alpha$$

(*ii*) $Cf \in R(\alpha)$ and

$$\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$$

(b) Define a rectifiable curve. If r' is continuous on $[a, b]$, then r is rectifiable and has length $\int_a^b |r'(t)|\, dt$.

7. (a) Define outer measure and show that it is countable subadditive and translation invariant.

(b) Show that the interval (a, b) is measurable.

8. (a) Let $\{E_i\}$ be an infinite decreasing sequence of measurable sets with $m^*(E_1) < \infty$. Then :

$$m^*\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{h\to\infty} m^*(E_n)$$

Also show that $m(E_1) < \infty$ is necessary in above theorem to hold true.

(b) Show that there exists a non-measurable set in the interval [0, 1).